Notes on Functional Analysis

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Contents

Metric Spaces and Normed Vector Spaces

Metric Space

A metric on a set M is a function $\rho : M \times M \to [0, \infty)$ such that

1. $\rho(x, y) = \rho(y, x)$, $\forall x, y \in M$.
2. $\rho(x, y) \le \rho(x, z) + \rho(z, y)$, $\forall x, y, z \in M$.
3. $\rho(x, y) = 0$ if and only if $x = y$.

Example

Let $M = \mathbb{R}^p$. Then, for $x = (x_1, \ldots, x_p) \in \mathbb{R}^p$ and $y = (y_1, \ldots, y_p) \in \mathbb{R}^p$,

$$\rho_2(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_p - y_p)^2},$$

$$\rho_1(x, y) = |x_1 - y_1| + \cdots + |x_p - y_p|, \quad \rho_\infty(x, y) = \max_{1 \le i \le p} |x_i - y_i|.$$

Convergence

$x_n \in M$ converges to $x \in M$ if $\rho(x_n, x) \xrightarrow{n \to \infty} 0$. We write $\lim_n x_n = x$ or $x_n \to x$ as $n \to \infty$.

Note. For the standard metrics on $\mathbb{R}^p$ $(\rho_1, \rho_2, \rho_\infty)$, $x^n = (x_1^n, \ldots, x_p^n) \to x = (x_1, \ldots, x_p)$ if and only if $x_i^n \to x_i$ for all $1 \le i \le p$.

Example

Let $M = C([0, 1])$, the space of continuous functions on $[0, 1]$. Then,

$$\rho_1(x, y) = \int_0^1 |x(t) - y(t)|\, dt, \quad \rho_\infty(x, y) = \sup_{0 \le t \le 1} |x(t) - y(t)|.$$

Let $K = \rho_\infty(x, y)$.

$$\rho_1(x, y) = \int_0^1 |x(t) - y(t)|\, dt \le \int_0^1 K\, dt = K = \rho_\infty(x, y).$$

Note that $\rho_1(x, y) \le \rho_\infty(x, y)$, so ρ_∞-convergence implies ρ_1-convergence, but not necessarily vice versa.

Cauchy Sequence

A sequence $x_n \in M$ is Cauchy if $\rho(x_n, x_m) \to 0$ as $n, m \to \infty$.

Proposition 1

If $x_n \to x$ in M, then (x_n) is Cauchy.

Proof. $\rho(x_n, x_m) \le \rho(x_n, x) + \rho(x_m, x) \to 0$ as $n, m \to \infty$

Completeness of Metric Space

(M, ρ) is complete if every Cauchy sequence has a limit. That is, if $\rho(x_n, x_m) \to 0$ as $n, m \to \infty$, then there exists $x \in M$ such that $x_n \to x$.

Example

a. $M = \mathbb{R}$ is complete, but $\mathbb{Q} \subset \mathbb{R}$ is not, with the subspace metric.

b. $M = L^1([0,1])$ with $\rho_1(x,y) = \int_0^1 |x(t) - y(t)|\, dt$ is complete.

c. $C([0,1]) \subset L^1([0,1])$ with $\rho_1(x,y)$ is not complete.

Remark. For example c, recall every $x \in L^1$ can be approximated by a sequence $x_n \in C([0,1])$. $\int_0^1 |x_n(t) - x(t)|\, dt \to 0$. Consider the specific sequence defined by the following step function:

$$x(t) = \begin{cases} 0 & \text{if } 0 \le t < \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} \le t \le 1 \end{cases} \in L^1([0,1]), \quad x_n(t) = \begin{cases} 0 & \text{if } 0 \le t < \frac{1}{2} \\ n(t - \frac{1}{2}) & \text{if } \frac{1}{2} \le t < \frac{1}{2} + \frac{1}{n} \\ 1 & \text{if } \frac{1}{2} + \frac{1}{n} \le t \le 1 \end{cases}.$$

Then,

$$\int_0^1 |x_n(t) - x(t)|\, dt = \frac{1}{2n} \to 0 \text{ as } n \to \infty.$$

x_n is Cauchy with respect to ρ_1 but $x \notin C([0,1])$.

Dense

$B \subseteq M$ is dense if every $x \in M$ can be approximated by $x_n \in B$: $x_n \in B \to x \in M$.

Theorem 1

For any (M, ρ) there exists a complete $(\tilde{M}, \tilde{\rho})$ such that M is isometric to a dense subset $B \subseteq \tilde{M}$. That is, there exists a bijection $h : M \to B$ such that

$$\rho(x, y) = \tilde{\rho}(h(x), h(y)), \quad x, y \in M.$$

$(\tilde{M}, \tilde{\rho})$ is called the completion of (M, ρ).

Remark. Isometric means distance-preserving. When we say that a map $h : M \to B$ is an isometry, it means that the distance between any two points in M is exactly equal to the distance between their corresponding images in B under the mapping h. In Theorem 1 above, h and its inverse $h^{-1} : B \to M$ are continuous.

Example

1. $\mathbb{R}$ is the completion of $\mathbb{Q}$.
2. $L^1([0,1])$ is the completion of $C([0,1])$ with respect to $\rho_1(x,y) = \int_0^1 |x(t) - y(t)|\, dt$.

Equivalence relation

If (x_n), (y_n) are Cauchy sequences, $(x_n) \sim (y_n)$ if $\rho(x_n, y_n) \xrightarrow{n\to\infty} 0$.

Continuous Functions

Let $f : (M, \rho) \to (Y, \nu)$ be a function.

1. f is continuous at x_0 if $x_n \to x_0$ implies $f(x_n) \to f(x_0)$. Equivalently, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\nu(f(x), f(x_0)) \leq \epsilon \text{ if } \rho(x_0, x) \leq \delta.$$

   We write $\lim_{x\to x_0} f(x) = f(x_0)$

2. f is continuous if f is continuous at all $x \in M$.

Open Balls and Closed Ball

$$B_\delta(x) = \{y \in M \mid \rho(x, y) < \delta\}, \quad B_\delta[x] = \{y \in M \mid \rho(x, y) \leq \delta\}$$

Open Set

$A \subseteq M$ is open if $\forall x \in A$, $\exists B_\delta(x) \subseteq A$ for some $\delta > 0$.

Closed Set

$F \subseteq M$ is closed if $F^c = M \setminus F$ is open.

Theorem 2

Let $\mathcal{T}$ be the collection of all open sets (all open subsets of M). Note that $M \in \mathcal{T}$, $\emptyset \in \mathcal{T}$. $\mathcal{T}$ is closed under arbitrary unions and finite intersections.

Remark. $\emptyset$ and M are open and closed. $\mathcal{T}$ is called the topology of M.

Corollary 1

Let $\mathcal{F}$ be the set of all closed sets. $\mathcal{F}$ is closed under arbitrary intersections and finite unions.

Neighborhood

$A \subseteq M$ is a neighborhood of x if $x \in A$ and there exists $B_\delta(x) \subseteq A$ with $\delta > 0$.

Inner Point

$x \in A$ is an inner point of A if there exists $B_\delta(x) \subseteq A$ with $\delta > 0$.

Fact. The set A° of all inner points of A is open.

Closure

Given $A \subseteq M$, its closure $\overline{A}$ is the smallest closed set that contains A.

Theorem 3

$\overline{A} = \{x \in M \mid \text{ there exists } \{x_n\} \subseteq A \text{ with } \lim_{n\to\infty} x_n = x\}$

Corollary 2

$F \subseteq M$ is closed iff $\overline{F} = F$ iff $\{x_n\} \subseteq F$ with $\lim_{n\to\infty} x_n = x$ implies $x \in F$.

Normed Vector Spaces (nvs)

A normed vector space E is a vector space over $\mathbb{R}$ (or $\mathbb{C}$) with a function, which we call a norm, $\|\cdot\|_E \colon E \to [0,\infty)$ such that

1. $\|\alpha x\|_E = |\alpha| \|x\|_E$
2. $\|x+y\|_E \le \|x\|_E + \|y\|_E$
3. $\|x\|_E = 0$ iff $x = 0$

Note. $\|\cdot\|_E$ defines the distance $\rho(x,y) = \|x-y\|_E$, $x, y \in E$.

Triangle inequality

$$\|x-y\|_E = \|(x-z)+(z-y)\|_E \le \|x-z\|_E + \|z-y\|_E.$$

Example

1. Examples of norms on $\mathbb{R}^d$ from which the previous metrics arise: for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$,
$$\|x\|_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2},$$
$$\|x\|_1 = |x_1| + \cdots + |x_d|,$$
$$\|x\|_\infty = \max_i |x_i|.$$
2. $E = C([0,1])$ with ρ_1, ρ_∞:
$$\|x\|_1 = \int_0^1 |x(t)|\,dt, \quad \|x\|_\infty = \sup_{0\le t\le 1} |x(t)|.$$

Completeness of Normed Vector Space

If a normed vector space E is complete, it is called Banach.

Example

1. $\mathbb{R}^d$ with $\|x\|_1, \|x\|_2, \|x\|_\infty$ is Banach.
2. $C[0,1]$ is Banach with sup norm $\|x\|_\infty$ but not Banach with $\|x\|_1$-norm.
3. $L^1[0,1]$ is Banach $\to \|x\|_1 = \int_0^1 |x(t)|\, dt$. $C[0,1]$ is dense in $L^1[0,1]$.
4. $\mathbb{Q}^d$ is not Banach.

Linear Operators

Let E, F be normed vector spaces, $T : E \to F$ be linear. The following claims are equivalent:

(a) T is continuous at 0.

(b) T is continuous on E.

(c) T is bounded: There exists $C > 0$ so that $\|T(x)\|_F \leq C\|x\|_E, \forall x \in E$.

Notation. We denote $\mathcal{L}(E, F)$ the set of all bounded linear $T : E \to F$

Proof. (b) $\implies$ (a) is obvious. For (a) $\implies$ (b), let $x_n \to x$ in E. Then, by linearity, $x_n - x \to 0$ in E and $T(x_n - x) = T(x_n) - T(x) \to 0$ as $n \to \infty$. Thus, (a) $\implies$ (b).

For (a) $\implies$ (c), since T is continuous at 0, there exists $\delta > 0$ such that

$$\|T(x)\|_F = \|T(x) - T(0)\|_F \leq 1 \text{ if } \|x\|_E = \|x - 0\|_E \leq \delta$$

For $y \in E$ and $y \neq 0$,

$$\|T(y)\|_F = \left\| T\left(\frac{y}{\|y\|_E} \cdot \delta\right) \right\|_F \cdot \frac{\|y\|_E}{\delta} \leq 1 \cdot \frac{\|y\|_E}{\delta} = \frac{1}{\delta} \cdot \|y\|_E = C\|y\|_E$$

where $C = 1/\delta$. For (c) $\implies$ (a), if $x_n \to 0$ then,

$$\|T(x_n)\|_F \leq C\|x_n\|_E \to 0 \text{ as } n \to \infty$$

Exercise. T is bounded iff $\sup \|T(x)\|_F < \infty$ for $\|x\|_E \leq R, \forall R > 0$

Operator Norm

Given $T \in \mathcal{L}(E, F)$, its operator norm is the number:

$$\|T\| = \sup_{\|x\|_E \leq 1} \|T(x)\|_F = \sup_{\|x\|_E = 1} \|T(x)\|_F$$

Note.

(i) $\|T\| \leq C$ in (c)

(ii) $\|T\|$ is the smallest $C > 0$ for which (c) holds

(iii) **Important:** $\mathbf{\|T(x)\|_F \leq \|T\|\|x\|_E}$

Exercise. $\mathcal{L}(E, F)$ with $\|\cdot\|$ is nvs.

Proof. We provide the proof for triangle inequality proof. Let $T, G \in \mathcal{L}(E, F)$, $\|x\|_E \leq 1$.

$$\sup_{\|x\|_E \leq 1} \|(T+G)(x)\|_F = \sup_{\|x\|_E \leq 1} \|T(x) + G(x)\|_F \leq \|T\| + \|G\|$$

Dual Space

Given nvs E, its **dual space** is $E^* = \mathcal{L}(E, \mathbb{R})$ (or $E^* = \mathcal{L}(E, \mathbb{C})$). Note that E^* is Banach. For $\ell \in E^*, x \in E$, we denote

$$\ell(x) = \langle \ell, x \rangle_{E^*, E} = \langle \ell, x \rangle$$

Example

1. $E = \mathbb{R}^d, \ell \in E^* = \mathcal{L}(\mathbb{R}^d, \mathbb{R})$ is $\ell(x) = a \cdot x = \langle a, x \rangle = \sum_{i=1}^d a_i x_i$ where $a = (a_1, \ldots, a_d) \in \mathbb{R}^d$, $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$.
2. Let E, F be nvs. Then $E \times F$ is nvs with $\|(x, y)\|_2 = \sqrt{\|x\|_E^2 + \|y\|_F^2}$. Also, $\|(x, y)\|_1 = \|x\|_E + \|y\|_F$, and $\|(x, y)\|_\infty = \max\{\|x\|_E, \|y\|_F\}$. Moreover, if E, F are Banach, then $E \times F$ is Banach.

Main Theorems of Normed Vector Spaces

Hahn-Banach Theorem

Let E be a normed vector space and $M \subseteq E$ be a vector subspace. Let $f \in M^* = \mathcal{L}(M, \mathbb{R})$ (the dual space of M). Then there exists $\ell \in E^* = \mathcal{L}(E, \mathbb{R})$ extending f such that $\|\ell\|_{E^*} = \|f\|_{M^*}$.

Remark. The proof uses Zorn's Lemma, but Zorn's Lemma is not needed if E is separable (i.e., if E has a countable dense subset).

Corollary 1

Let E be a normed vector space and $F \subset E$ be a closed vector subspace such that $F \neq E$. Then, there exists a nonzero $\ell \in E^*$ so that $\ell|_F = 0$ (i.e., $\ell(y) = 0$ for all $y \in F$). Moreover, if $x_0 \notin F$, then there exists $\ell \in E^*$ so that $\ell(x_0) = 1$ and $\ell(y) = 0$ for any $y \in F$.

Proof. Let $x_0 \notin F$. We begin by constructing the subspace M:

$$M = \mathbb{R}x_0 + F = \{tx_0 + y : t \in \mathbb{R}, y \in F\}.$$

Before proceeding, we establish a crucial property: if $tx_0 + y = 0$, then $t = 0$ and $y = 0$. To prove this, suppose $tx_0 + y = 0$ with $t \neq 0$. Then we would have $x_0 = -\frac{y}{t} \in F$ since F is a subspace, contradicting our assumption that $x_0 \notin F$. Therefore, t must be 0, and consequently $y = 0$.

We now define a linear functional $f : M \to \mathbb{R}$ by $f(tx_0 + y) = t$. Note that this definition implies $f(x_0) = 1$ and $f(y) = 0$ for all $y \in F$. To show f is well-defined, suppose $tx_0 + y = t'x_0 + y'$. Then $(t - t')x_0 + (y - y') = 0$. By our previous observation, this implies $t = t'$ and $y = y'$, so $f(tx_0 + y) = f(t'x_0 + y')$. The linearity of f follows naturally:

$$f(\alpha_1(t_1x_0 + y_1) + \alpha_2(t_2x_0 + y_2)) = \alpha_1t_1 + \alpha_2t_2 = \alpha_1 f(t_1x_0 + y_1) + \alpha_2 f(t_2x_0 + y_2).$$

To prove f is bounded, let $d = \inf_{z\in F} \|x_0 - z\|_E$. We claim that $d > 0$. Since F is closed and $x_0 \notin F$, if we had $d = 0$, there would exist a sequence $\{z_n\} \subseteq F$ with $z_n \to x_0$, contradicting $x_0 \notin F$.

For any $tx_0 + y \in M$, we have

$$\|tx_0 + y\|_E = |t| \left\|x_0 + \frac{y}{t}\right\|_E = |t| \left\|x_0 - \left(-\frac{y}{t}\right)\right\|_E \geq |t|d.$$

Therefore, if $\|tx_0 + y\|_E \leq 1$, then $|t| \leq \frac{1}{d}$. This shows:

$$\|f\|_{M^*} = \sup_{\|tx_0+y\|_E\leq 1} |f(tx_0 + y)| = \sup_{\|tx_0+y\|_E\leq 1} |t| \leq \frac{1}{d} < \infty.$$

By the Hahn-Banach Theorem, f extends to a functional $\ell \in E^*$. This extended functional ℓ satisfies $\ell(x_0) = f(x_0) = 1$ and $\ell(y) = f(y) = 0$ for all $y \in F$, completing our proof.

Comment. Corollary 1 is a standard tool to prove by contradiction that a vector subspace $D \subseteq E$ is dense ($\overline{D} = E$).

1. Assume the opposite, $\overline{D} \neq E$. Note that $\overline{D}$ is a closed vector subspace.
2. By Corollary 1, there exists a nonzero $\ell \in E^*$ such that $\ell|_{\overline{D}} = 0$.
3. But if other arguments show that $\ell|_{\overline{D}} = 0 \implies \ell = 0$ on E, we reach a contradiction.

Note that if $\ell|_D = 0$ and ℓ is continuous, then for any sequence $\{x_n\} \subseteq D$ with $x_n \to x_0 \in \overline{D}$, we have $\ell(x_n) \to \ell(x_0) \implies \ell(x_0) = 0$.

Corollary 2

(a) For each nonzero $x \in E$, there exists a nonzero $\ell \in E^*$ such that

$$\ell(x) = \langle \ell, x\rangle = \|\ell\|_{E^*}\|x\|_E.$$

We say ℓ and x are aligned.

(b) For any $x \in E$,

$$\|x\|_E = \sup_{\|\ell\|_{E^*}\leq 1} |\ell(x)|.$$

Recall that $\|\ell\|_{E^*} = \sup_{\|x\|_E\leq 1} |\ell(x)|$.

Remark. It is always the case that $|\ell(x)| \leq \|\ell\|_{E^*}\|x\|_E$. For (a):

1. If $E = \mathbb{R}^d$ with $\|\cdot\|_2$, then $E^* = \mathbb{R}^d$ with $\|\cdot\|_2$. Two vectors $\ell = (a_1, \ldots, a_d)$ and $x = (x_1, \ldots, x_d)$ are aligned if

$$\ell(x) = \langle a, x\rangle = a_1x_1 + \cdots + a_dx_d = \|a\|_2\|x\|_2.$$

2. If $E = \mathbb{R}^d$ with $\|\cdot\|_1$, then $E^* = \mathbb{R}^d$ with $\|\cdot\|_\infty$, and we have:

$$\langle a, x\rangle = \|a\|_\infty\|x\|_1.$$

Proof. Assume the base field is $\mathbb{R}$.

(a) Let $x_0 \in E$ and $x_0 \neq 0$. Define

$$M = \mathbb{R}x_0 = \{tx_0 : t \in \mathbb{R}\}.$$

Define a linear map $f : M \to \mathbb{R}$ by $f(tx_0) = t\|x_0\|_E$, and then:

$$\|f\|_{M^*} = \sup_{\|tx_0\|_E \leq 1} |f(tx_0)| = \sup_{|t|\|x_0\|_E \leq 1} |t|\|x_0\|_E = 1.$$

Note that $f(x_0) = \|x_0\|_E = \|f\|_{M^*}\|x_0\|_E$. Thus, f is linear and bounded, so $f \in \mathcal{L}(M, \mathbb{R})$. By the Hahn-Banach Theorem, there exists $\ell \in E^*$ extending f with

$$\|\ell\|_{E^*} = \|f\|_{M^*} = 1,$$

$$\implies \ell(x_0) = f(x_0) = \|x_0\|_E = \|\ell\|_{E^*}\|x_0\|_E.$$

(b) Let $x_0 \in E$ and $\|\ell\|_{E^*} \leq 1$. Then,

$$|\ell(x_0)| \leq \|\ell\|_{E^*}\|x_0\|_E \leq \|x_0\|_E \implies \sup_{\|\ell\|_{E^*} \leq 1} |\ell(x_0)| \leq \|x_0\|_E.$$

By part (a), there exists $\ell \in E^*$ such that $\|\ell\|_{E^*} = 1$ and $\ell(x_0) = \|x_0\|_E$.

Norm of Adjoint Operator

Let E, F be normed vector spaces and $T \in \mathcal{L}(E, F)$. We define the adjoint $T' : F^* \to E^*$ to be the linear map that assigns to each $\ell \in F^*$ the functional $T'\ell \in E^*$ defined by

$$(T'\ell)(x) := \ell(T(x)), \quad x \in E.$$

Proposition 1

$T' \in \mathcal{L}(F^*, E^*)$ and $\|T'\| = \|T\|$.

Proof.

$$\|T'\| = \sup_{\|\ell\|_{F^*} \leq 1} \|T'\ell\|_{E^*} = \sup_{\|\ell\|_{F^*} \leq 1} \sup_{\|x\|_E \leq 1} |(T'\ell)(x)| = \sup_{\|\ell\|_{F^*} \leq 1} \sup_{\|x\|_E \leq 1} |\ell(T(x))|$$

$$\implies \|T'\| = \sup_{\|\ell\|_{F^*} \leq 1} \sup_{\|x\|_E \leq 1} |\ell(T(x))| = \sup_{\|x\|_E \leq 1} \sup_{\|\ell\|_{F^*} \leq 1} |\ell(T(x))|$$

By Corollary 2 (b),

$$\|T'\| = \sup_{\|x\|_E \leq 1} \sup_{\|\ell\|_{F^*} \leq 1} |\ell(T(x))| = \sup_{\|x\|_E \leq 1} \|T(x)\|_F = \|T\|$$

Embedding

Let E be a normed vector space and $E^{**} = (E^*)^* = \mathcal{L}(E^*, \mathbb{R})$. Every $x_0 \in E$ defines naturally a functional J_{x_0} defined by

$$J_{x_0}(\ell) = \ell(x_0), \quad \ell \in E^*$$

Note that $|J_{x_0}(\ell)| = |\ell(x_0)| \leq \|\ell\|_{E^*}\|x_0\|_E$.

Proposition 2

The map $J : E \to E^{**}$ that assigns to $x_0 \in E$ the linear functional $J_{x_0} \in E^{**}$ is an isometry:

$$\|x_0\|_E = \|J_{x_0}\|_{E^{**}}$$

i.e., the space E is isometrically embedded into E^{**} and we write $E \subseteq E^{**}$.

Proof. For $x_0 \in E$, by Corollary 2 (b),

$$\|x_0\|_E = \sup_{\|\ell\|_{E^*} \leq 1} |\ell(x_0)| = \sup_{\|\ell\|_{E^*} \leq 1} |J_{x_0}(\ell)| = \|J_{x_0}\|_{E^{**}}$$

Baire's Lemma

Let (M, ρ) be a complete metric space and $M = \bigcup_n M_n$ with all M_n closed. Then, there exists $n_0 \geq 1$ so that $\mathring{M}_{n_0} \neq \emptyset$, i.e., there exists $B_{r_0}(x_0) \subseteq M_{n_0}$. Recall $\mathring{M}_{n_0} = \text{Int}(M_{n_0}) = \{x \in M_{n_0} : B_r(x) \subseteq M_{n_0} \text{ for some } r > 0\}$.

Proof. We prove by contradiction. Assume $\mathring{M}_n = \emptyset$, $\forall n \geq 1$. Then for each n,

$$O_n = M_n^c = M \setminus M_n \text{ is dense and open in } M$$

Every $B_r(x) \subseteq M$ contains $y \in O_n$. We will show that $G = \bigcap_n O_n$ is dense, but

$$G = \bigcap_n O_n = \bigcap_n M_n^c = \left(\bigcup_n M_n\right)^c = M^c = \emptyset$$

and this is a contradiction.

To see G is dense, note G is dense if every $B_{r_0}(x_0)$ contains $x \in G$. Fix $B_{r_0}(x_0)$. Since O_1 is dense,

$$\exists \overline{B}_{r_1}(x_1) \subseteq B_{r_0}(x_0) \cap O_1, \text{ with } r_1 \leq \frac{r_0}{2}$$

To get $\overline{B}_{r_1}(x_1)$, we can always find an open ball in a nonempty open set. Thus, we can take a closed ball to be the closure of an open ball with half the radius (or the closure of any other smaller ball).

Since O_2 is dense, $B_{r_1}(x_1) \cap O_2$ is nonempty and open.

$$\exists \overline{B}_{r_2}(x_2) \subseteq B_{r_1}(x_1) \cap O_2 \text{ with } r_2 \leq \frac{r_1}{2} \leq \frac{r_0}{2^2}$$

Since O_{n+1} is dense,

$$\exists \overline{B}_{r_{n+1}}(x_{n+1}) \subseteq B_{r_n}(x_n) \cap O_{n+1} \text{ with } r_{n+1} \leq \frac{r_0}{2^{n+1}}$$

We claim the sequence $\{x_n\}$ is Cauchy. If $j, j' \geq n$, then $x_j, x_{j'} \in B_{r_n}(x_n)$, and

$$\rho(x_j, x_{j'}) \leq \rho(x_j, x_n) + \rho(x_{j'}, x_n) \leq 2 \cdot \frac{r_0}{2^n} \to 0 \text{ as } j, j' \to \infty$$

Since M is complete, $x = \lim_{n\to\infty} x_n$ exists. Note that

$$x \in \bigcap_{n\geq 1} \overline{B}_{r_n}(x_n) \subseteq \bigcap_{n\geq 0} B_{r_n}(x_n) \subseteq O_m, \quad \forall m \geq 1$$

because $\overline{B}_{r_n}(x_n)$ is closed so the limit belongs to it. This means $x \in \bigcap_{m\geq 1} O_m = G$.

Banach-Steinhaus Theorem

Theorem 2

Let E, F be normed vector spaces and E be Banach. Let $T_i \in \mathcal{L}(E, F)$, $i \in I$, and for each $x \in E$,

$$\sup_{i\in I} \|T_i(x)\|_F < \infty \text{ (pointwise uniform boundedness)}$$

Then $\sup_{i\in I} \|T_i\| < \infty$, or equivalently, for every $R > 0$,

$$\sup_{i\in I} \sup_{\|x\|_E \leq R} \|T_i(x)\|_F = \sup_{\|x\|_E \leq R} \sup_{i\in I} \|T_i(x)\|_F < \infty$$

Proof. By assumption, for each $n \geq 1$, the set

$$M_n = \{x \in E : \sup_{i\in I} \|T_i(x)\|_F \leq n\} = \bigcap_{i\in I} \{x \in E : \|T_i(x)\|_F \leq n\}$$

is closed because each T_i is continuous. Since every $x \in E$ is pointwise bounded, we have $E = \bigcup_{n=1}^{\infty} M_n$. By Baire's Lemma, there exists $n_0 \geq 1$ such that $\mathring{M}_{n_0} \neq \emptyset$, meaning:

$$\exists \overline{B}_{r_0}(x_0) \subseteq \{x \in E : \sup_{i\in I} \|T_i(x)\|_F \leq n_0\}$$

We will show that

$$\|T_i\| = \frac{1}{r_0} \sup_{\|y\|_E \leq r_0} \|T_i(y)\|_F \leq \frac{1}{r_0} C_0$$

where C_0 is independent of i, by showing:

$$\sup_{\|y\|_E \leq r_0} \|T_i(y)\|_F \leq C_0$$

Let $y \in E$ with $\|y\|_E \leq r_0$. Then,

$$\|T_i(y)\|_F = \|T_i(y + x_0 - x_0)\|_F \leq \|T_i(x_0 + y)\|_F + \|T_i(x_0)\|_F \leq n_0 + \sup_{i\in I} \|T_i(x_0)\|_F = 2n_0$$

because $x_0 + y \in \overline{B}_{r_0}(x_0)$ and $\sup_{i\in I} \|T_i(x_0)\|_F \le n_0$. Thus, we can set $C_0 = 2n_0$.

Remark. Let $T_i \in \mathcal{L}(\mathbb{R}^d, \mathbb{R}^d)$ be given by matrix components:

$$T_i = (a^i_{k\ell}), \quad 1 \le k \le d,\, 1 \le \ell \le d$$

Then $\sup_i \|T_i(x)\| < \infty$ for all $x \in \mathbb{R}^d$ implies $\sup_i |a^i_{k\ell}| < \infty$ for all k, ℓ. Indeed, if e_k $(k = 1, \dots, d)$ is the standard basis of $\mathbb{R}^d$, then

$$\sup_i |a^i_{k\ell}| = \sup_i |e_k \cdot (T_i(e_\ell))| \le \sup_i \|T_i(e_\ell)\| < \infty, \quad \ell = 1, \dots, d$$

As $\|e_k\| = 1$ and $|\cos\alpha| \le 1$, we have:

$$e_k \cdot (T_i(e_\ell)) = \|e_k\| \|T_i(e_\ell)\| \cos\alpha$$

Open Mapping Theorem

Let M, Y be metric spaces and $f : M \to Y$. The following are equivalent:

(a) f is continuous

(b) $f^{-1}(U) \subseteq M$ is open for any open $U \subseteq Y$

(c) $f^{-1}(F) \subseteq M$ is closed for any closed $F \subseteq Y$

Let E, F be Banach spaces and $T \in \mathcal{L}(E, F)$ be surjective (onto). If $U \subseteq E$ is open, then $T(U) \subseteq F$ is open.

Theorem 3

Let E, F be Banach spaces and $T \in \mathcal{L}(E, F)$ be surjective. Then there exists $c > 0$ so that $B^F_c \subseteq T(B^E_1)$ where

$$B^F_c = \{y \in F : \|y\|_F < c\}, \quad B^E_1 = \{x \in E : \|x\|_E < 1\}$$

Here, $B^H_r = \{x \in H : \|x\|_H < r\}$ denotes a general ball of radius r in space H.

Proof. Since T is onto,

$$E = \bigcup_{n=1}^{\infty} B^E_n, \quad F = \bigcup_{n=1}^{\infty} \overline{T(B^E_n)}$$

We take the closure here to apply Baire's Lemma. By Baire's Lemma, there exists $n_0 \ge 1$ such that $B^F_{r_0}(x_0) \subseteq \overline{T(B^E_{n_0})}$. Since T is linear, we have $y = T(v) \implies -y = T(-v)$, meaning $y \in T(B^E_{n_0}) \implies -y \in T(B^E_{n_0})$. Thus, $B^F_{r_0}(-x_0) \subseteq \overline{T(B^E_{n_0})}$.

For $y \in B^F_{r_0}$,

$$y = \frac{1}{2}(x_0 + y) + \frac{1}{2}(-x_0 + y) \in \overline{T(B^E_{n_0})}$$

where $x_0 + y \in B^F_{r_0}(x_0)$ and $-x_0 + y \in B^F_{r_0}(-x_0)$. By convexity, $B^F_{r_0} \subseteq \overline{T(B^E_{n_0})}$. Now we claim:

$$B^F_{r_0/n_0} \subseteq \overline{T(B^E_1)} \subseteq T(B^E_2)$$

Let $\delta = r_0/n_0$. If this holds, then $B_\delta^F \subseteq T(B_2^E) \implies B_{\delta/2}^F \subseteq T(B_1^E)$.

We first show $\overline{T(B_1^E)} \subseteq T(B_2^E)$. Let $z \in \overline{T(B_1^E)}$. Then there exists $x_1 \in B_1^E$ such that

$$z - T(x_1) \in B_{\delta/2}^F \subseteq \overline{T(B_{1/2}^E)}$$

This is because z lies in the closure of the image of B_1^E, allowing us to approximate z closely by elements in the image. Similarly, there exists $x_2 \in B_{1/2}^E$ such that:

$$z - T(x_1) - T(x_2) \in B_{\delta/2^2}^F \subseteq \overline{T(B_{1/2^2}^E)}$$

Inductively, there exists $x_n \in B_{1/2^{n-1}}^E$ such that:

$$z - T(x_1 + \cdots + x_n) \in B_{\delta/2^n}^F \subseteq \overline{T(B_{1/2^n}^E)}$$

Note that:

$$\|z - T(x_1 + \cdots + x_n)\|_F \leq \frac{\delta}{2^n} \xrightarrow{n\to\infty} 0$$

The series $\sum_{k=1}^{\infty} x_k$ converges absolutely since:

$$\sum_{k=1}^{\infty} \|x_k\|_E \leq 1 + \sum_{k=1}^{\infty} \frac{1}{2^k} = 2$$

Since E is Banach, $x = \sum_{k=1}^{\infty} x_k = \lim_{n\to\infty} \sum_{k=1}^{n} x_k$ exists, and:

$$\|x\|_E \leq \sum_{k=1}^{\infty} \|x_k\|_E < 2$$

Therefore,

$$z = \lim_{n\to\infty} T(x_1 + \cdots + x_n) = T(x) \quad \text{and} \quad \|x\|_E < 2$$

Hence, $z \in T(B_2^E)$, which proves $\overline{T(B_1^E)} \subseteq T(B_2^E)$.

Corollary 3

Let E, F be Banach spaces and $T \in \mathcal{L}(E, F)$ be onto. Then, $T(U) \subseteq F$ is open for any open $U \subseteq E$.

Proof. Let $U \subseteq E$ be open and $y_0 \in T(U)$. Then, there exists $x_0 \in U$ and $\delta > 0$ so that $B_\delta^E(x_0) \subseteq U$ and $y_0 = T(x_0)$. By Theorem 3, there exists $c > 0$ such that $B_c^F \subseteq T(B_1^E)$, implying $B_{c\delta}^F \subseteq T(B_\delta^E)$. We have:

$$y_0 + B_{c\delta}^F \subseteq T(x_0 + B_\delta^E) = T(B_\delta^E(x_0)) \subseteq T(U)$$

Therefore, $T(U)$ is an open set.

Inverse Mapping Theorem

Let E, F be Banach spaces and $T \in \mathcal{L}(E, F)$ be bijective. Then, $T^{-1} \in \mathcal{L}(F, E)$.

Proof. Let $U \subseteq E$ be open. Then $(T^{-1})^{-1}(U) = T(U)$ is open by the Open Mapping Theorem. Thus, T^{-1} is continuous, so it is bounded.

Closed Graph Theorem

Let E, F be Banach spaces and $T : E \to F$ be linear. Assume its graph:

$$\Gamma = \{(x, T(x)) \in E \times F : x \in E\} \subseteq E \times F \text{ is closed.}$$

Then, $T \in \mathcal{L}(E, F)$.

Proof. Since Γ is a closed vector subspace of $E \times F$, Γ is a Banach space as well. Both projections of Γ are continuous:

$$(x, T(x)) \in \Gamma \xrightarrow{\pi_1} x \in E$$

$$(x, T(x)) \in \Gamma \xrightarrow{\pi_2} T(x) \in F$$

Since $\|\pi_1(x, T(x))\|_E = \|x\|_E \leq \sqrt{\|x\|_E^2 + \|T(x)\|_F^2} = \|(x, T(x))\|_{E \times F}$, π_1 is bounded and clearly bijective. By the Inverse Mapping Theorem, $\pi_1^{-1} : E \to \Gamma$ is continuous. Since $T = \pi_2 \circ \pi_1^{-1}$ is the composition of two continuous functions, T is continuous and thus bounded.

Comment. Proving continuity of $T : E \to F$:

1. In a standard way, given $x_n \to x$ in E:
   (i) Prove $\lim_n T(x_n) = y$ exists.
   (ii) Prove $y = T(x)$.
2. Using the Closed Graph Theorem: Given $x_n \to x$ in E and $T(x_n) \to y$ in F, prove $y = T(x)$.

Hilbert Spaces

Inner Product

Let H be a complex vector space. An inner product on H is a $\mathbb{C}$-valued function denoted by $\langle x, y \rangle$, for $x, y \in H$, such that:

i. $\langle x, x \rangle \geq 0 \; \forall x \in H$; $\langle x, x \rangle = 0$ if and only if $x = 0$

ii. $\forall x, y, z \in H$, and $\alpha, \beta \in \mathbb{C}$:

$$\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle,$$

$$\langle z, \alpha x + \beta y \rangle = \overline{\alpha} \langle z, x \rangle + \overline{\beta} \langle z, y \rangle,$$

$$\langle x, y \rangle = \overline{\langle y, x \rangle}.$$

Examples

1. $H = \mathbb{C}^d$, $x = (x_1, \ldots, x_d)$, $y = (y_1, \ldots, y_d) \in \mathbb{C}^d$:

$$\langle x, y\rangle = \sum_{j=1}^{d} x_j \overline{y}_j.$$

   Then, $\langle x, x\rangle = \sum_{j=1}^{d} |x_j|^2$.

2. ℓ^2 is the space of all sequences $(x_1, x_2, \ldots)$ such that

$$\sum_{n=1}^{\infty} |x_n|^2 < \infty,$$

   with the inner product $\langle x, y\rangle = \sum_{n=1}^{\infty} x_n \overline{y}_n$.

3. $\mathcal{C} = C([a, b])$ is the space of continuous complex-valued functions on $[a, b]$ with

$$\langle f, g\rangle = \int_a^b f(t)\overline{g(t)}\, dt.$$

   Then, $\langle f, f\rangle = \int_a^b |f(t)|^2 dt$.

4. $L^2(X, \mu)$ is the space of complex-valued square-integrable functions (equivalence classes) on a measure space $(X, \mathcal{X}, \mu)$ with

$$\langle f, g\rangle = \int f(x)\overline{g(x)}\, d\mu.$$

   In fact, Examples 1 and 2 are just specific cases of Example 4 using the counting measure.

Inner Product Space (Pre-Hilbert Space)

A complex vector space equipped with an inner product is called an inner product space or pre-Hilbert space.

Orthogonality

Let H be a pre-Hilbert space. We say $x \perp y$ in H if $\langle x, y\rangle = 0$. The induced norm is defined as:

$$\|x\|_H := \sqrt{\langle x, x\rangle}, \quad x \in H.$$

Pythagorean Theorem

If $x \perp y$, then

$$\|x + y\|_H^2 = \|x\|_H^2 + \|y\|_H^2.$$

Proof. Let $\langle x, y\rangle = \langle y, x\rangle = 0$. Then,

$$\begin{aligned}\|x+y\|_H^2 &= \langle x+y, x+y\rangle \\ &= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \\ &= \|x\|_H^2 + \|y\|_H^2\end{aligned}$$

Orthonormal System

A subset $\{e_j : j = 1, \dots, N\}$ is an orthonormal system (ONS) if

$$\langle e_k, e_\ell\rangle = \begin{cases} 1 & \text{if } k = \ell \\ 0 & \text{if } k \neq \ell \end{cases}.$$

Note that $\|e_k\|_H^2 = \langle e_k, e_k\rangle = 1$.

Theorem 2

Let $\{e_j : j = 1, \dots, N\}$ be an ONS. Then for any $x \in H$:

i. $\sum_{j=1}^N \langle x, e_j\rangle e_j$ and $x - \sum_{j=1}^N \langle x, e_j\rangle e_j$ are orthogonal.

ii. $\|x\|_H^2 = \|x - \sum_{j=1}^N \langle x, e_j\rangle e_j\|_H^2 + \|\sum_{j=1}^N \langle x, e_j\rangle e_j\|_H^2$.

iii. **Bessel's Inequality:**

$$\sum_{j=1}^N |\langle x, e_j\rangle|^2 \leq \|x\|_H^2.$$

Proof. We first prove (i) by showing that $x - \sum_{j=1}^N \langle x, e_j\rangle e_j \perp e_k$ for all $k = 1, \dots, N$:

$$\langle x - \sum_{j=1}^N \langle x, e_j\rangle e_j, e_k\rangle = \langle x, e_k\rangle - \langle x, e_k\rangle\langle e_k, e_k\rangle = \langle x, e_k\rangle - \langle x, e_k\rangle = 0$$

Next, (ii) follows directly from (i) and the Pythagorean Theorem, and (iii) clearly follows by dropping the non-negative first term from the right-hand side of (ii).

Remark. If $\{e_j : j = 1, \dots, N\}$ is an ONS, then $\{e_1, \dots, e_N\}$ is linearly independent. Indeed,

$$0 = \sum_{j=1}^N \alpha_j e_j \implies \alpha_k = 0 \quad \forall k$$

which is shown by taking the inner product of both sides with each e_k.

Cauchy-Schwarz Inequality

For any $x, y \in H$,

$$|\langle x, y\rangle| \leq \|x\|_H \|y\|_H$$

Proof. Let $y \neq 0$. Consider the single-element ONS $\{\frac{y}{\|y\|_H}\}$. By Bessel's inequality:

$$\sum_{j=1}^{1} |\langle x, e_j\rangle|^2 \leq \|x\|_H^2 \implies \left|\left\langle x, \frac{y}{\|y\|_H}\right\rangle\right|^2 \leq \|x\|_H^2 \implies |\langle x, y\rangle|^2 \leq \|x\|_H^2 \|y\|_H^2$$

Corollary 1

$\|x\|_H = \sqrt{\langle x, x\rangle}$, $x \in H$, is a well-defined norm.

Proof. We prove the triangle inequality. For any $x, y \in H$:

$$\begin{aligned}
\|x + y\|_H^2 &= \langle x + y, x + y\rangle \\
&= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \\
&\leq \langle x, x\rangle + |\langle x, y\rangle| + |\langle y, x\rangle| + \langle y, y\rangle \\
&\leq \|x\|_H^2 + \|y\|_H^2 + 2\|x\|_H \|y\|_H \quad \text{(by Cauchy-Schwarz)} \\
&= (\|x\|_H + \|y\|_H)^2
\end{aligned}$$

Hilbert Space

A pre-Hilbert space H is called a **Hilbert space** if it is complete with respect to the induced norm $\|\cdot\|_H$.

Projection Theorem

Let H be a pre-Hilbert space. Given a vector subspace $M \subseteq H$, its orthogonal complement is defined as:

$$M^\perp = \{x \in H : \langle x, m\rangle = 0 \quad \forall m \in M\}$$

We write $x \perp M$ if $x \in M^\perp$. Note that $M^\perp$ is always a closed vector subspace of H due to the continuity of the inner product.

Theorem 3

Let H be a Hilbert space and $M \subseteq H$ be a closed vector subspace.

i. $\forall x \in H$, there exists a unique $m_0 \in M$ such that

$$d := \inf_{m \in M} \|x - m\|_H = \|x - m_0\|_H$$

Moreover, $(x - m_0) \perp M$. The vector m_0 is called the **orthogonal projection** of x onto M, denoted by $P_M x = m_0$.

ii. Let $m_1 \in M$ be such that $(x - m_1) \perp M$. Then $m_1 = m_0$.

Remark. Theorem 3 implies that $\forall x \in H$, there exists a unique decomposition $x = m_0 + w$ where $m_0 \in M$ and $w \in M^\perp$. Thus, $H = M \oplus M^\perp$.

Proof of i. Let $x \in H$. By the definition of infimum, there exists a sequence $\{m_k\} \subseteq M$ such that $\|x - m_k\|_H \to d$ as $k \to \infty$. We show that $\{m_k\}$ is a Cauchy sequence using the Parallelogram Law:

$$\|m_k - m_\ell\|_H^2 = \|(x - m_\ell) - (x - m_k)\|_H^2 = 2\|x - m_\ell\|_H^2 + 2\|x - m_k\|_H^2 - \|2x - (m_\ell + m_k)\|_H^2$$

Since M is a vector subspace, $\frac{m_\ell + m_k}{2} \in M$, which implies $\|x - (\frac{m_\ell + m_k}{2})\|_H \geq d$. Thus,

$$\|2x - (m_\ell + m_k)\|_H^2 = 4\left\|x - \left(\frac{m_\ell + m_k}{2}\right)\right\|_H^2 \geq 4d^2$$

Substituting this back gives:

$$\|m_k - m_\ell\|_H^2 \leq 2\|x - m_\ell\|_H^2 + 2\|x - m_k\|_H^2 - 4d^2 \xrightarrow{k,\ell \to \infty} 2d^2 + 2d^2 - 4d^2 = 0$$

Since H is complete, $m_0 := \lim_{k \to \infty} m_k$ exists, and by closure of M, $m_0 \in M$ with $\|x - m_0\|_H = d$.

For uniqueness, if $\tilde{m}_0 \in M$ also achieves the distance d, applying the same inequality with $m_k = m_0$ and $m_\ell = \tilde{m}_0$ yields:

$$\|m_0 - \tilde{m}_0\|_H^2 \leq 2d^2 + 2d^2 - 4d^2 = 0 \implies m_0 = \tilde{m}_0$$

To prove that $(x - m_0) \perp M$, let $m \in M$ and $t \in \mathbb{R}$. Expanding the norm using inner product properties:

$$\|(x - m_0) - tm\|_H^2 = \|x - m_0\|_H^2 - 2t\mathrm{Re}\{\langle x - m_0, m\rangle\} + t^2\|m\|_H^2$$

Since m_0 minimizes the distance, we have $d^2 \leq \|(x - m_0) - tm\|_H^2$, which simplifies to:

$$0 \leq t^2\|m\|_H^2 - 2t\mathrm{Re}\{\langle x - m_0, m\rangle\} \quad \forall t \in \mathbb{R}$$

For this quadratic function to remain non-negative for all t, the linear coefficient must vanish:

$$\mathrm{Re}\{\langle x - m_0, m\rangle\} = 0$$

Replacing t by it in the original formulation yields $\mathrm{Im}\{\langle x - m_0, m\rangle\} = 0$. Since both parts are zero, $\langle x - m_0, m\rangle = 0$, proving orthogonality.

Proof of ii. Let $m_1 \in M$ satisfy $(x - m_1) \perp M$. For any $m \in M$, since $(m_1 - m) \in M$, we have $(x - m_1) \perp (m_1 - m)$. By the Pythagorean Theorem:

$$\|x - m\|_H^2 = \|(x - m_1) + (m_1 - m)\|_H^2 = \|x - m_1\|_H^2 + \|m_1 - m\|_H^2 \geq \|x - m_1\|_H^2$$

Taking the infimum over all $m \in M$ implies $\|x - m_1\|_H^2 = d^2$. By the uniqueness proven in part (i), $m_1 = m_0$.

Corollary 2

Let $\{e_1, \ldots, e_N\}$ be an ONS in a Hilbert space H, and let $M = \text{span}\{e_1, \ldots, e_N\}$. Then for any $x \in H$:

$$P_M(x) = \sum_{j=1}^{N} \langle x, e_j \rangle e_j$$

Proof. Theorem 2 established that $(x - \sum_{j=1}^{N} \langle x, e_j \rangle e_j) \perp e_k$ for all k, which implies it is orthogonal to the entire span M. Applying Theorem 3 (ii) completes the proof.

Hilbert Space Basis

Let H be a Hilbert space. An orthonormal system $S \subseteq H$ is called a **complete orthonormal system (CONS)** or a **Hilbert basis** if for any $z \in H$:

$$\langle z, e_\alpha \rangle = 0 \quad \forall e_\alpha \in S \implies z = 0$$

Example

Let μ be the counting measure on $[0, 1]$. Consider $H = L^2([0, 1], \mu)$, the space of functions $f : [0, 1] \to \mathbb{R}$ such that

$$\sum_{t \in [0,1]} [f(t)]^2 = \int_{[0,1]} [f(t)]^2 d\mu < \infty$$

The inner product is given by $\langle f, g \rangle = \sum_{t \in [0,1]} f(t)g(t)$. For each $\alpha \in [0, 1]$, defining the standard indicator functions $e_\alpha(t) = 1$ if $\alpha = t$ and 0 otherwise forms a non-countable CONS where $f(t) = \sum_{\alpha \in [0,1]} \langle f, e_\alpha \rangle e_\alpha(t)$.

Theorem 4

Let H be a Hilbert space with a CONS $\{e_\alpha : \alpha \in A\}$. For any $x \in H$:

i. The set $\mathcal{A} := \{\alpha \in A : \langle x, e_\alpha \rangle \neq 0\}$ is at most countable: $\mathcal{A} = \{\alpha_1, \alpha_2, \ldots\}$.

ii. $x = \sum_{\alpha \in A} \langle x, e_\alpha \rangle e_\alpha = \sum_{k=1}^{\infty} \langle x, e_{\alpha_k} \rangle e_{\alpha_k} = \lim_{n \to \infty} \sum_{k=1}^{n} \langle x, e_{\alpha_k} \rangle e_{\alpha_k}$.

iii. **Parseval's Identity:**

$$\|x\|_H^2 = \sum_{\alpha \in A} |\langle x, e_\alpha \rangle|^2 = \sum_{k=1}^{\infty} |\langle x, e_{\alpha_k} \rangle|^2$$

Proof.

i. For any finite subset $\{\beta_1, \ldots, \beta_n\} \subseteq A$, Bessel's inequality states $\sum_{k=1}^{n} |\langle x, e_{\beta_k} \rangle|^2 \leq \|x\|_H^2 < \infty$. For each $n \geq 1$, define $A_n = \{\alpha \in A : |\langle x, e_\alpha \rangle| > \frac{1}{n}\}$. If any A_n were infinite, it would contain a countable sequence contradicting Bessel's bounded sum. Hence, each A_n is finite, making $\mathcal{A} = \bigcup_{n=1}^{\infty} A_n$ a countable union of finite sets, which is countable.

ii. Let $\{\alpha_k\}_{k=1}^{\infty}$ be an enumeration of $\mathcal{A}$ and define the partial sums $s_n = \sum_{k=1}^{n} \langle x, e_{\alpha_k} \rangle e_{\alpha_k}$. For $m > n$, by the Pythagorean theorem:

$$\|s_m - s_n\|_H^2 = \sum_{k=n+1}^{m} |\langle x, e_{\alpha_k} \rangle|^2$$

As $n, m \to \infty$, this goes to 0 because the total series $\sum_{k=1}^{\infty} |\langle x, e_{\alpha_k} \rangle|^2$ converges. Thus $\{s_n\}$ is Cauchy, and since H is complete, it converges to some $y \in H$. To show $y = x$, we look at the inner product for any $\alpha \in A$:

$$\langle x - y, e_\alpha \rangle = \langle x, e_\alpha \rangle - \lim_{n\to\infty} \sum_{k=1}^{n} \langle x, e_{\alpha_k} \rangle \langle e_{\alpha_k}, e_\alpha \rangle$$

If $\alpha \notin \mathcal{A}$, both terms are 0. If $\alpha \in \mathcal{A}$, then $\alpha = \alpha_n$ for some n, yielding $\langle x, e_{\alpha_n} \rangle - \langle x, e_{\alpha_n} \rangle = 0$. Since $\langle x - y, e_\alpha \rangle = 0$ for all α, by the completeness of the system, $x - y = 0 \implies x = y$.

iii. Applying Theorem 2 (ii) to the finite partial sums yields:

$$\|x\|_H^2 = \left\| x - \sum_{k=1}^{n} \langle x, e_{\alpha_k} \rangle e_{\alpha_k} \right\|_H^2 + \sum_{k=1}^{n} |\langle x, e_{\alpha_k} \rangle|^2$$

Taking $n \to \infty$, the first norm term vanishes due to part (ii), leaving Parseval's Identity.

Theorem 5

A Hilbert space H has a countable CONS if and only if H is separable (contains a countable dense subset).

Proof. ($\Longrightarrow$) Let $\{e_n\}_{n\geq 1}$ be a countable CONS. The set of all finite linear combinations $\sum_{k=1}^{n} r_k e_k$ where $r_k \in \mathbb{Q} + i\mathbb{Q}$ forms a countable dense subset in H. ($\Longleftarrow$) Let $\{y_1, y_2, \ldots\}$ be a countable dense subset. We can filter out linearly dependent vectors to get a linearly independent sequence $\{x_n\}$ spanning the same subspace. Applying the standard Gram-Schmidt procedure to $\{x_n\}$ constructs a countable orthonormal system $\{e_n\}$ whose span is dense in H, hence forming a CONS.

Proposition 2

Let H be a pre-Hilbert space. For a fixed $z \in H$, define $\ell_z(x) = \langle x, z \rangle$. Then $\ell_z \in H^*$ and $\|\ell_z\|_{H^*} = \|z\|_H$.

Proof. Boundedness follows from Cauchy-Schwarz: $|\ell_z(x)| = |\langle x, z \rangle| \leq \|z\|_H \|x\|_H$, showing $\|\ell_z\|_{H^*} \leq \|z\|_H$. Testing the functional on the unit vector $x = \frac{z}{\|z\|_H}$ (for $z \neq 0$) yields exactly $\|z\|_H$, establishing equality.

Riesz Representation Theorem

Let H be a Hilbert space. For every bounded linear functional $\ell \in H^*$, there exists a unique $z_\ell \in H$ such that

$$\ell(x) = \langle x, z_\ell \rangle, \quad \forall x \in H$$

Moreover, $\|\ell\|_{H^*} = \|z_\ell\|_H$.

Proof. If $\ell = 0$, choosing $z_\ell = 0$ works. Assume $\ell \neq 0$, and let $M = \ker(\ell)$. Since ℓ is continuous, M is a closed subspace, and $M \neq H$. By the Projection Theorem, $M^\perp \neq \{0\}$. Pick a nonzero $x_0 \in M^\perp$ scaled such that $\ell(x_0) = 1$. For any $x \in H$, the vector $x - \ell(x)x_0$ belongs to M because $\ell(x - \ell(x)x_0) = \ell(x) - \ell(x) = 0$. Since $x_0 \in M^\perp$, we have:

$$\langle x - \ell(x)x_0, x_0 \rangle = 0 \implies \langle x, x_0 \rangle = \ell(x)\|x_0\|_H^2 \implies \ell(x) = \left\langle x, \frac{x_0}{\|x_0\|_H^2} \right\rangle$$

Setting $z_\ell = \frac{x_0}{\|x_0\|_H^2}$ proves existence. Uniqueness follows since if $\langle x, z_1 \rangle = \langle x, z_2 \rangle$ for all x, then $\langle x, z_1 - z_2 \rangle = 0 \implies z_1 - z_2 = 0$.

Adjoint Operator

Let H be a Hilbert space. For each $T \in \mathcal{L}(H)$, there exists a unique $T^* \in \mathcal{L}(H)$ such that

$$\langle Tx, y \rangle = \langle x, T^*y \rangle \quad \forall x, y \in H$$

Moreover, $\|T^*\| = \|T\|$.

Proof. For a fixed $y \in H$, $x \mapsto \langle Tx, y \rangle$ is a bounded linear functional on H. By the Riesz Representation Theorem, there exists a unique vector, which we denote as T^*y, such that $\langle Tx, y \rangle = \langle x, T^*y \rangle$. To verify that T^* is linear on a complex Hilbert space, consider $y_1, y_2 \in H$ and $\alpha, \beta \in \mathbb{C}$. For any $x \in H$:

$$\begin{aligned}
\langle x, T^*(\alpha y_1 + \beta y_2) \rangle &= \langle Tx, \alpha y_1 + \beta y_2 \rangle \\
&= \overline{\alpha}\langle Tx, y_1 \rangle + \overline{\beta}\langle Tx, y_2 \rangle \quad \text{(conjugate-linear in 2nd slot)} \\
&= \overline{\alpha}\langle x, T^*y_1 \rangle + \overline{\beta}\langle x, T^*y_2 \rangle \\
&= \langle x, \alpha T^*y_1 + \beta T^*y_2 \rangle \quad \text{(bringing scalars back inside)}
\end{aligned}$$

Since this holds for all $x \in H$, $T^*(\alpha y_1 + \beta y_2) = \alpha T^*y_1 + \beta T^*y_2$. The norm equality follows from:

$$\|T^*\| = \sup_{\|y\|\leq 1} \sup_{\|x\|\leq 1} |\langle x, T^*y \rangle| = \sup_{\|y\|\leq 1} \sup_{\|x\|\leq 1} |\langle Tx, y \rangle| = \sup_{\|x\|\leq 1} \|Tx\|_H = \|T\|$$

Self-Adjoint Operator

An operator $T \in \mathcal{L}(H)$ is called **self-adjoint** if $T^* = T$.

Lemma

Let H be a Hilbert space and $T \in \mathcal{L}(H)$. Then,

$$\|T^*T\| = \|T\|^2 = \|T^*\|^2$$

Proof. For any $x \in H$ with $\|x\|_H \leq 1$:

$$\|Tx\|_H^2 = \langle Tx, Tx\rangle = \langle x, T^*Tx\rangle \leq \|x\|_H \|T^*Tx\|_H \leq \|T^*T\|$$

Taking the supremum over all such x yields $\|T\|^2 \leq \|T^*T\|$. Combined with the standard norm inequality $\|T^*T\| \leq \|T^*\|\|T\| = \|T\|^2$, we get equality.

Remark

Let H be a Hilbert space. Then:

1. $\mathcal{A} := \mathcal{L}(H)$ is a C^*-algebra.
2. $\mathcal{A}$ is a Banach algebra satisfying $\|ST\| \leq \|S\|\|T\|$.
3. The adjoint operation $T \mapsto T^*$ is an involution satisfying:
   - $(\alpha T + \beta S)^* = \overline{\alpha} T^* + \overline{\beta} S^*$
   - $(ST)^* = T^* S^*$
   - $\|T^*T\| = \|T\|^2$

L^p Spaces

Let $(X, \mathcal{X}, \mu)$ be a measure space. For $0 < p \leq \infty$, we define the space $L^p(X, \mathcal{X}, \mu)$ (abbreviated as $L^p(\mu)$ or L^p) as:

$$L^p = \left\{ f : X \to \mathbb{C} \text{ is measurable} : \int_X |f|^p \, d\mu < \infty \right\}.$$

We identify functions that are equal almost everywhere (a.e.) to form equivalence classes. Additionally, we define L^∞ to be the space of equivalence classes of measurable functions $f : X \to \mathbb{C}$ that are essentially bounded:

$$|f| \leq C \text{ a.e. for some } C > 0.$$

Remark. L^p is a vector space. For instance, if $p \in (0, \infty)$ and $a, b > 0$, the inequality $(a+b)^p \leq 2^p(a^p + b^p)$ holds because:

$$(a+b)^p \leq (2\max\{a,b\})^p = 2^p \max\{a^p, b^p\} \leq 2^p(a^p + b^p).$$

For $f \in L^p$ where $p \in (0, \infty)$, we define the L^p norm (or quasi-norm if $p < 1$) as:

$$\|f\|_{L^p} = \|f\|_p = \left(\int_X |f|^p \, d\mu \right)^{1/p}.$$

For $p = \infty$, the essential supremum norm is defined as:

$$\|f\|_{L^\infty} = \|f\|_\infty = \inf\{C \geq 0 : |f| \leq C \text{ a.e.}\}.$$

Note. By definition, $|f| \leq \|f\|_\infty$ a.e.

Young's Inequality

For $a, b \geq 0$, $p \in (1, \infty)$, and $\frac{1}{p} + \frac{1}{q} = 1$,

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Proof. Let $a, b > 0$ (the case where $a = 0$ or $b = 0$ is trivial). Since the natural logarithm function ln is concave, we can apply Jensen's inequality:

$$\ln\left(\frac{a^p}{p} + \frac{b^q}{q}\right) \geq \frac{1}{p}\ln(a^p) + \frac{1}{q}\ln(b^q) = \ln(a) + \ln(b) = \ln(ab).$$

Taking the exponential (exp) of both sides yields the desired result.

Hölder's Inequality

Let $p \in [1, \infty]$ and let $\frac{1}{p} + \frac{1}{q} = 1$ (with $\frac{1}{\infty} = 0$). If $f \in L^p$ and $g \in L^q$, then $fg \in L^1$ and:

$$\int_X |fg|\, d\mu \leq \|f\|_p \|g\|_q$$

.

Proof. For $p = 1$ or $p = \infty$, the proof is straightforward and left as an exercise. Let $p \in (1, \infty)$, $f \neq 0$, and $g \neq 0$ a.e. By Young's inequality, for any parameter $t > 0$, we have:

$$\int_X |fg|\, d\mu = \int_X |(tf)(t^{-1}g)|\, d\mu \leq \frac{t^p}{p}\int_X |f|^p\, d\mu + \frac{t^{-q}}{q}\int_X |g|^q\, d\mu$$

. Let $A = \int_X |f|^p\, d\mu$ and $B = \int_X |g|^q\, d\mu$. Define the auxiliary function $L(t)$ for $t > 0$:

$$L(t) = \frac{t^p}{p}A + \frac{t^{-q}}{q}B$$

. To minimize $L(t)$, we find its derivative and set it to zero:

$$L'(t) = t^{p-1}A - t^{-q-1}B = 0 \implies t^{p+q} = \frac{B}{A} \implies t = \left(\frac{B}{A}\right)^{\frac{1}{pq}}$$

Evaluating $L(t)$ at this critical point gives:

$$\inf_{t>0} L(t) = \frac{1}{p}\left(\frac{B}{A}\right)^{1/q} A + \frac{1}{q}\left(\frac{B}{A}\right)^{-1/p} B = \frac{1}{p}A^{1/p}B^{1/q} + \frac{1}{q}A^{1/p}B^{1/q} = A^{1/p}B^{1/q}$$

Substituting $A^{1/p} = \|f\|_p$ and $B^{1/q} = \|g\|_q$ completes the proof.

Remark. If $f \in L^p$ and $\frac{1}{p} + \frac{1}{q} = 1$, then $|f|^{p-1} \in L^q$.

Proposition 1 (Minkowski's Inequality)

For $p \in [1, \infty]$, $\| \cdot \|_p$ satisfies the triangle inequality and forms a norm on L^p.

Proof. For $p = 1$ or $p = \infty$, the proof is left as an exercise. Let $p \in (1, \infty)$ and $f, g \in L^p$. We assume $f + g \neq 0$ dynamically. Splitting the integrand yields:

$$\int_X |f+g|^p \, d\mu = \int_X |f+g|^{p-1}|f+g| \, d\mu \leq \int_X |f+g|^{p-1}|f| \, d\mu + \int_X |f+g|^{p-1}|g| \, d\mu$$

Applying Hölder's inequality to both integrals on the right-hand side using conjugate exponents p and $q = \frac{p}{p-1}$, we get:

$$\int_X |f+g|^p \, d\mu \leq \left(\int_X |f+g|^{(p-1)q} \, d\mu \right)^{1/q} \|f\|_p + \left(\int_X |f+g|^{(p-1)q} \, d\mu \right)^{1/q} \|g\|_p$$

Since $(p-1)q = p$ and $1 - \frac{1}{q} = \frac{1}{p}$, this simplifies directly to:

$$\int_X |f+g|^p \, d\mu \leq \left(\int_X |f+g|^p \, d\mu \right)^{1-\frac{1}{p}} (\|f\|_p + \|g\|_p)$$

Dividing both sides by $\left(\int_X |f+g|^p \, d\mu \right)^{1-\frac{1}{p}}$ yields:

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p$$

Remark. Let $f_n, f \in L^\infty$. Then $\|f_n - f\|_\infty \to 0$ if and only if there exist representatives $\tilde{f}_n, \tilde{f}$ in their respective equivalence classes such that $\sup_{x \in X} |\tilde{f}_n(x) - \tilde{f}(x)| \to 0$ as $n \to \infty$.

Chebyshev's Inequality

Let $p \in (0, \infty)$ and $f \in L^p$. Then, for any $\epsilon > 0$:

$$\mu(\{x \in X : |f(x)| > \epsilon\}) \leq \frac{\int_X |f|^p \, d\mu}{\epsilon^p}$$

Proof. Let $\epsilon > 0$. By utilizing monotonic sets, we observe:

$$\int_X |f|^p \, d\mu \geq \int_{\{|f| > \epsilon\}} |f|^p \, d\mu \geq \epsilon^p \mu(\{x \in X : |f(x)| > \epsilon\})$$

Theorem 2

L^p is a Banach space for any $p \in [1, \infty]$.

Proof. The case $p = \infty$ is left as an exercise. Let $p \in [1, \infty)$ and let (f_n) be a Cauchy sequence in L^p. By Chebyshev's Inequality, for any $\epsilon > 0$:

$$\mu(\{|f_n - f_m| > \epsilon\}) \leq \frac{\int_X |f_n - f_m|^p \, d\mu}{\epsilon^p} \xrightarrow{n,m \to \infty} 0$$

Hence, (f_n) is Cauchy in measure. This implies there exists a measurable function f and a subsequence (f_{n_k}) such that $f_{n_k} \to f$ a.e. as $k \to \infty$.

Fix $\epsilon > 0$. Choose N such that $\int_X |f_n - f_m|^p \, d\mu \le \epsilon$ for all $n, m \ge N$. Letting $m = n_l \to \infty$, Fatou's Lemma implies:

$$\int_X |f_n - f|^p \, d\mu = \int_X \lim_{l \to \infty} |f_n - f_{n_l}|^p \, d\mu \le \liminf_{l \to \infty} \int_X |f_n - f_{n_l}|^p \, d\mu \le \epsilon$$

for all $n \ge N$. Thus, $f - f_n \in L^p$, which means $f = (f - f_n) + f_n \in L^p$, and $f_n \to f$ in L^p.

Comparison of L^p Spaces

In general, spaces are not nested: neither $L^p \subseteq L^q$ nor $L^q \subseteq L^p$ if $p \ne q$ on arbitrary measure spaces.

Proof. Let $X = (0, \infty)$ equipped with Lebesgue measure μ. Consider the function:

$$f(x) = x^{-\alpha} \chi_{(0,1)}(x) + x^{-\beta} \chi_{[1,\infty)}(x) \quad (\alpha, \beta > 0)$$

Calculating the L^r integral gives $\int_X |f(x)|^r \, d\mu = \int_0^1 x^{-\alpha r} \, dx + \int_1^\infty x^{-\beta r} \, dx$. This converges if and only if $\alpha < \frac{1}{r} < \beta$. Thus, if we select exponents such that $\frac{1}{q} < \alpha < \frac{1}{p} < \beta$, then $f \in L^p$ but $f \notin L^q$. Conversely, choosing $\alpha < \frac{1}{q} < \beta < \frac{1}{p}$ yields $f \in L^q$ but $f \notin L^p$.

Proposition 3

Let $\mu(X) < \infty$ and $0 < p < q \le \infty$. Then $L^q \subseteq L^p$ and:

$$\|f\|_p \le \|f\|_q \, \mu(X)^{\frac{1}{p} - \frac{1}{q}}$$

Proof. Applying Hölder's inequality with conjugate exponents $\frac{q}{p}$ and $\frac{q}{q-p}$ yields:

$$\int_X |f|^p \cdot 1 \, d\mu \le \left(\int_X |f|^q \, d\mu \right)^{\frac{p}{q}} \left(\int_X 1 \, d\mu \right)^{1 - \frac{p}{q}} = \|f\|_q^p \, \mu(X)^{1 - \frac{p}{q}}$$

Taking the p-th root of both sides proves the assertion.

Proposition 4

Let $0 < p < q < r \le \infty$. Then $L^q \subseteq L^p + L^r$.

Proof. Let $f \in L^q$. We decompose $f = f\chi_{\{|f| \le 1\}} + f\chi_{\{|f| > 1\}} =: k + h$. Then $|k| \le |f|$ and $|k| \le 1$, so $|k|^r \le |f|^q \in L^1$, implying $k \in L^r$. Similarly, $|h| \le |f|$ and on the support of h, $|f| > 1$, meaning $|h|^p \le |f|^q \in L^1$, implying $h \in L^p$.

Proposition 5

Let $0 < p < q < r \leq \infty$. Then $L^p \cap L^r \subseteq L^q$ and $\|f\|_q \leq \|f\|_p^t \|f\|_r^{1-t}$ where $\frac{1}{q} = \frac{t}{p} + \frac{1-t}{r}$.

Proof. Let $f \in L^p \cap L^r$.

i. If $r = \infty$, then $\int_X |f|^q \, d\mu = \int_X |f|^{q-p} |f|^p \, d\mu \leq \|f\|_\infty^{q-p} \|f\|_p^p$. Taking the q-th root gives the result with $t = p/q$.

ii. If $r < \infty$, rewrite $|f|^q = |f|^{tq} |f|^{(1-t)q}$. Apply Hölder's inequality with conjugate exponents $\frac{p}{tq}$ and $\frac{r}{(1-t)q}$. The conditions require $\frac{tq}{p} + \frac{(1-t)q}{r} = 1$, which aligns perfectly with the definition of t.

Proposition 6 (Sequence Spaces)

If $X = \mathbb{N}$ and μ is the counting measure, we denote $\ell^p = L^p(\mathbb{N}, \mu)$. If $0 < p < q \leq \infty$, then $\ell^p \subseteq \ell^q$ and $\|x\|_q \leq \|x\|_p$.

Proof. Let $x \in \ell^p$. For $q = \infty$, $\|x\|_\infty^p = \sup_n |x_n|^p \leq \sum_n |x_n|^p = \|x\|_p^p \implies \|x\|_\infty \leq \|x\|_p$. For $q < \infty$, by interpolation (Proposition 5), $\|x\|_q \leq \|x\|_p^{p/q} \|x\|_\infty^{1-p/q} \leq \|x\|_p$.

Simple Function Spaces

Define $\Sigma = \{f \in \mathcal{E} : f$ vanishes outside a set of finite measure$\}$, where $\mathcal{E}$ is the space of all complex-valued simple functions:

$$\mathcal{E} = \left\{ f = \sum_{j=1}^{n} z_j \chi_{E_j} : n \geq 1,\, z_j \in \mathbb{C},\, E_j \in \mathcal{X} \text{ are disjoint} \right\}$$

Remark. $\mathcal{E} \subseteq L^\infty$ and $\Sigma \subseteq L^p$ for any $p \in (0, \infty]$.

Theorem 2.10 [Folland]

Let $(X, \mathcal{X})$ be a measurable space. If $f : X \to \mathbb{C}$ is measurable, there exists a sequence $\{f_n\} \subseteq \mathcal{E}$ such that $0 \leq |f_1| \leq |f_2| \leq \cdots \leq |f|$, $f_n \to f$ pointwise, and $f_n \to f$ uniformly on any set where f is bounded.

Proposition 7

i. $\mathcal{E}$ is dense in L^∞.

ii. Σ is dense in L^p if $p \in [1, \infty)$.

Proof. Apply Theorem 2.10. For (i), since $f \in L^\infty$ is bounded outside a null set, $f_n \to f$ uniformly a.e., meaning $\|f_n - f\|_\infty \to 0$. For (ii), if $f \in L^p$, $|f_n - f|^p \leq 2^{p+1} |f|^p \in L^1$. By the Dominated Convergence Theorem, $\int_X |f_n - f|^p \, d\mu \to 0$.

Generalized Dominated Convergence Theorem

Let $\{f_n\}$ be a sequence of measurable functions such that $f_n \to f$ a.e. Suppose there exists a sequence of nonnegative integrable functions $\{g_n\}$ such that $|f_n| \leq g_n$ a.e., $g_n \to g$ a.e., and $\int_X g_n \, d\mu \to \int_X g \, d\mu < \infty$. Then $f_n \to f$ in L^1.

Proof. Consider the sequence of nonnegative functions $h_n = g_n + g - |f_n - f|$. Since $|f_n - f| \leq g_n + g$ a.e., $h_n \geq 0$ a.e. Applying Fatou's Lemma to $\{h_n\}$ yields:

$$\int_X \liminf_{n\to\infty} h_n \, d\mu \leq \liminf_{n\to\infty} \int_X h_n \, d\mu$$

Since $f_n \to f$ and $g_n \to g$ a.e., $\liminf h_n = 2g$ a.e. Thus:

$$\int_X 2g \, d\mu \leq \int_X g \, d\mu + \int_X g \, d\mu - \limsup_{n\to\infty} \int_X |f_n - f| \, d\mu$$

Subtracting $2\int_X g \, d\mu < \infty$ from both sides results in $\limsup_{n\to\infty} \int_X |f_n - f| \, d\mu \leq 0$, which ensures $\lim_{n\to\infty} \|f_n - f\|_1 = 0$.

The Dual of L^p

Let $p \in (1, \infty]$ and $q \in [1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$. For any $h \in L^q$, define the linear functional T_h on L^p by:

$$T_h(f) = \int_X fh \, d\mu, \quad f \in L^p$$

Then $T_h \in (L^p)^*$ and $\|T_h\|_{(L^p)^*} = \|h\|_q = \sup\{|\int_X fh \, d\mu| : \|f\|_p \leq 1\}$.

Theorem 4 (Riesz Representation Theorem for L^p)

Let $p \in (1, \infty)$. For each bounded linear functional $T \in (L^p)^*$, there exists a unique $h \in L^q$ such that $T(f) = \int_X hf \, d\mu$ for all $f \in L^p$, and $\|T\|_{(L^p)^*} = \|h\|_q$. If μ is σ-finite, this also holds for $p = 1$ and $q = \infty$.

Weak L^p

Let $p \in (0, \infty)$. A measurable function f belongs to **weak** L^p if its distribution function $\lambda_f(\alpha) = \mu(\{x \in X : |f(x)| > \alpha\})$ satisfies:

$$[f]_p := \left(\sup_{\alpha > 0} \alpha^p \lambda_f(\alpha)\right)^{1/p} < \infty$$

1. By Chebyshev's Inequality, $L^p \subseteq$ weak L^p and $[f]_p \leq \|f\|_p$.
2. Using layer-cake integration by parts, if $f \in L^p$, its norm can be computed via:

$$\|f\|_p^p = \int_X |f|^p \, d\mu = p \int_0^\infty \alpha^{p-1} \lambda_f(\alpha) \, d\alpha$$

Convolution and Smoothing

Let $X = \mathbb{R}^d$, $\mathcal{X} = \mathcal{L}$, and $\mu = m \implies d\mu = dm = dx$.

Lemma 1

Let f, g be measurable, $x_0 \in \mathbb{R}^d$. Then,

$$f(x_0 - \cdot)g \in L^1 \iff g(x_0 - \cdot)f \in L^1 \tag{1}$$

and if Eq.(2) holds,

$$\int f(x_0 - y)g(y)\,dy = \int g(x_0 - y)f(y)\,dy,$$

All functions involved in Eq.(2) are measurable including $f(x_0 - \cdot)$ and $g(x_0 - \cdot)$.

Proof. Let $G : \mathbb{R}^d \to \mathbb{R}^d$ be bijective with $G, G^{-1} \in C^1$ and $x = G(y) = (g_1(y), \ldots, g_d(y))$. Note that C^1 denotes the class of continuously differentiable functions. Then, by the formula of changing variables,

$$\int f(x)\,dx = \int f(G(y))|\det \nabla G(y)|\,dy.$$

$$\nabla G(y) = \begin{bmatrix} \frac{\partial g_1(y)}{\partial y_1} & \cdots & \frac{\partial g_1(y)}{\partial y_d} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_d(y)}{\partial y_1} & \cdots & \frac{\partial g_d(y)}{\partial y_d} \end{bmatrix}.$$

In our case, we have $G(y) = x_0 - y = z$. Then,

$$\nabla G(y) = \begin{bmatrix} -1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 \end{bmatrix} = -I_d \implies |\det \nabla G(y)| = |(-1)^d| = 1.$$

$$\implies \int f(x_0 - y)g(y)\,dy = \int f(G(y))g(x_0 - G(y))|\det \nabla G(y)|\,dy = \int g(x_0 - z)f(z)\,dz.$$

Convolution

Let f, g be measurable and $f(x - \cdot)g \in L^1$ a.e. with respect to x. The following function is well-defined:

$$(f * g)(x) = \int f(x - y)g(y)\,dy = \int g(x - y)f(y)\,dy = (g * f)(x).$$

$f * g$ is called the convolution of f and g.

Well-defined Convolution

Let f, g be measurable. If

$$\int |f(x-y)||g(y)|\,dy < \infty \quad \text{a.e. with respect to } x,$$

then we say

$$(f * g)(x) = \int f(x-y)g(y)\,dy,$$

is well-defined. We say $(f * g)(x)$ is defined for the x such that

$$\int |f(x-y)||g(y)|\,dy < \infty.$$

Support

Given $f : \mathbb{R}^d \to \mathbb{C}$, the support of f, denoted $\text{supp}(f)$, is the closure of $\{x : f(x) \neq 0\}$. $C_c(\mathbb{R}^d)$ is the space of continuous functions with compact support.

Lemma 2

If $f, g \in C_c(\mathbb{R}^d)$, then $f * g \in C_c(\mathbb{R}^d)$. Moreover,

$$\text{supp}(f * g) \subseteq \overline{\text{supp}(f) + \text{supp}(g)},$$

where $A + B = \{x + y : x \in A, y \in B\}$.

Proof. Since $f, g \in C_c(\mathbb{R}^d)$, both f, g are bounded. Therefore,

$$\int |f(x-y)||g(y)|\,dy < \infty \text{ for any } x \in \mathbb{R}^d.$$

Hence, $f*g$ is defined for any $x \in \mathbb{R}^d$. Now, let $\{x_n\}$ be a sequence that converges to $x_0 \in \mathbb{R}^d$. Define $h_n(y) = [f(x_n - y) - f(x_0 - y)]g(y)$ and $h(y) = [f(x_0 - y) - f(x_0 - y)]g(y) = 0$. Next, we apply the Dominated Convergence Theorem to prove $f * g$ is continuous. DCT needs:

1. $h_n \to h$ a.e.
2. There exists a nonnegative $\phi \in L^1$ with $|h_n| \leq \phi$ a.e. for all n.

Since $f \in C_c(\mathbb{R}^d)$, f is continuous. Therefore, for each fixed y, we have $f(x_n - y) \to f(x_0 - y)$ as $x_n \to x_0$. This gives us pointwise convergence: $h_n(y) \to 0 = h(y)$ for each y. We need an integrable function that dominates $|h_n(y)|$. Note that since f has compact support and is continuous, it's bounded: $|f(z)| \leq M$ for some constant M. Then,

$$|h_n(y)| = |[f(x_n - y) - f(x_0 - y)]g(y)| \leq 2M|g(y)|$$

Since $g \in C_c(\mathbb{R}^d)$, g is bounded, so $\phi = 2M|g(y)| \in L^1$. By DCT,

$$\lim_{n\to\infty} \left(\int f(x_n - y)g(y)\,dy - \int f(x_0 - y)g(y)\,dy \right) = \lim_{n\to\infty} \int h_n(y)\,dy = \int \lim_{n\to\infty} h_n(y)\,dy = 0$$

Therefore, $f * g$ is continuous.

Now, we claim $\{x : (f * g)(x) \neq 0\} \subseteq \text{supp}(f) + \text{supp}(g)$.

$$\begin{aligned}(f * g)(x) \neq 0 &\implies \text{There exists } y \text{ such that } f(x-y)g(y) \neq 0 \\ &\implies x - y \in \text{supp}(f), \quad y \in \text{supp}(g) \\ &\implies x = (x-y) + y \in \text{supp}(f) + \text{supp}(g) \\ &\implies \{x : (f * g)(x) \neq 0\} \subseteq \text{supp}(f) + \text{supp}(g)\end{aligned}$$

Taking the closure of both sides,

$$\text{supp}(f * g) = \overline{\{x : (f * g)(x) \neq 0\}} \subseteq \overline{\text{supp}(f) + \text{supp}(g)}$$

Since both $\text{supp}(f)$ and $\text{supp}(g)$ are compact sets, we use a theorem from topology: The Minkowski sum of two compact sets is compact. Thus, $\text{supp}(f) + \text{supp}(g)$ is compact. Since it is compact, it is closed in $\mathbb{R}^d$. Therefore,

$$\overline{\text{supp}(f) + \text{supp}(g)} = \text{supp}(f) + \text{supp}(g)$$

This gives us $\text{supp}(f * g) \subseteq \text{supp}(f) + \text{supp}(g)$. Since $\text{supp}(f * g)$ is a closed subset of the compact set $\text{supp}(f) + \text{supp}(g)$, it is compact. Hence, $f * g$ is a continuous function with compact support, so $f * g \in C_c(\mathbb{R}^d)$.

Remark.

1. If a function has compact support, it must be bounded, and here's why: A set in $\mathbb{R}^d$ is compact if and only if it is closed and bounded. So when we say a function f has compact support, it means that the closure of the set $\{x : f(x) \neq 0\}$ is both closed and bounded. For a continuous function f with compact support, we can apply the extreme value theorem, which states that a continuous function on a compact set attains both its maximum and minimum values. Specifically:

   - Since f is continuous and its support is compact
   - By the extreme value theorem, f must attain a maximum value on its support
   - Outside its support, f is zero by definition
   - Therefore, f is bounded by the maximum absolute value it achieves on its support

   This means that there exists some constant $M > 0$ such that $|f(x)| \leq M$ for all $x \in \mathbb{R}^d$.

2. Lebesgue measure on $\mathbb{R}^d$ is indeed sigma-finite. We can express $\mathbb{R}^d$ as a countable union of bounded sets:

   $$\mathbb{R}^d = \bigcup_{n=1}^{\infty} B_n$$

   where $B_n = \{x \in \mathbb{R}^d : |x| \leq n\}$ is the closed ball of radius n centered at the origin. Each B_n has finite Lebesgue measure. Specifically:

   $$\lambda(B_n) = \frac{\pi^{d/2}}{\Gamma(d/2+1)} \cdot n^d < \infty$$

This decomposition into a countable union of finite-measure sets is exactly the definition of a sigma-finite measure space.

Proposition 1

Let $p \in [1, \infty]$, $\frac{1}{p} + \frac{1}{q} = 1$, $f \in L^p$, $h \in L^q$. Then, $(f * h)(x)$ is defined for all $x \in \mathbb{R}^d$ and

$$\sup_x |(f * h)(x)| \leq \|f\|_p \|h\|_q$$

Moreover, $f * h$ is continuous. Also, $\lim_{|x| \to \infty} (f * h)(x) = 0$ if $p \in (1, \infty)$.

Proof. For each $x \in \mathbb{R}^d$, by Hölder's inequality,

$$\int |f(x-y)||h(y)|\, dy \leq \|f(x - \cdot)\|_p \|h\|_q = \|f\|_p \|h\|_q < \infty$$

$$\implies |(f * h)(x)| = \left| \int f(x-y)h(y)\, dy \right| \leq \int |f(x-y)||h(y)|\, dy \leq \|f\|_p \|h\|_q$$
$$\implies \sup_x |(f * h)(x)| \leq \|f\|_p \|h\|_q$$

Continuity

(i) Let $p \in (1, \infty)$. There exist $f_n, h_n \in C_c(\mathbb{R}^d)$ such that $\|f_n - f\|_p \to 0$ and $\|h_n - h\|_q \to 0$ as $n \to \infty$. Recall by Lemma 2, $f_n * h_n \in C_c(\mathbb{R}^d)$, and

$$\begin{aligned} \sup_x |(f_n * h_n)(x) - (f * h)(x)| &\leq \sup_x |((f_n - f) * h_n)(x)| + \sup_x |(f * (h_n - h))(x)| \\ &\leq \|f_n - f\|_p \|h_n\|_q + \|f\|_p \|h_n - h\|_q \\ &\leq \left(\sup_k \|h_k\|_q \right) \|f_n - f\|_p + \|f\|_p \|h_n - h\|_q \xrightarrow{n \to \infty} 0 \end{aligned}$$

Thus, $f_n * h_n \to f * h$ uniformly. The uniform convergence of a sequence of continuous functions preserves continuity, so $f * h$ must be continuous. Also,

$$\lim_{|x| \to \infty} (f_n * h_n)(x) = 0 \implies \lim_{|x| \to \infty} (f * h)(x) = 0$$

(ii) Let $p = 1$, $f \in L^1$, $h \in L^\infty$. There exists $f_n \in C_c(\mathbb{R}^d)$ such that $\|f_n - f\|_1 \xrightarrow{n \to \infty} 0$. Then,

$$\sup_x |(f_n * h)(x) - (f * h)(x)| = \sup_x |((f_n - f) * h)(x)| \leq \|f_n - f\|_1 \|h\|_\infty \xrightarrow{n \to \infty} 0$$

Thus, $f_n * h$ converges uniformly to $f * h$ as $n \to \infty$. So we just need to show that $(f_n * h)(x) = \int f_n(x-y)h(y)\, dy$ is continuous in x. By the DCT and continuity of f_n:

$$\lim_{x \to x_0} ((f_n * h)(x) - (f_n * h)(x_0)) = \int \lim_{x \to x_0} [f_n(x-y) - f_n(x_0 - y)]h(y) = 0$$

The uniform convergence of a sequence of continuous functions preserves continuity, so $f * h$ must be continuous.

Remark. The property $\lim_{|x|\to\infty}(f_n * h_n)(x) = 0$ for functions $f_n * h_n \in C_c(\mathbb{R}^d)$ follows directly from the definition of compact support. By definition, a function $g \in C_c(\mathbb{R}^d)$ has compact support, meaning that the set $\text{supp}(g) = \overline{\{x \in \mathbb{R}^d : g(x) \neq 0\}}$ is compact. Since $\text{supp}(g)$ is compact in $\mathbb{R}^d$, it must be bounded. This means there exists some radius $R > 0$ such that:

$$\text{supp}(g) \subseteq B(0, R) = \{x \in \mathbb{R}^d : |x| \leq R\}$$

In other words, for any $|x| > R$, we have $x \notin \text{supp}(g)$, which means $g(x) = 0$. Applying this to $f_n * h_n \in C_c(\mathbb{R}^d)$, there exists some $R_n > 0$ such that for all $|x| > R_n$, we have $(f_n * h_n)(x) = 0$. Therefore, $\lim_{|x|\to\infty}(f_n * h_n)(x) = 0$ is immediate because the function is exactly zero outside some bounded region.

Proposition 2

Let $p \in [1, \infty)$, $f \in L^p$, $h \in L^1$. Then, $f * h$ is well-defined and $f * h \in L^p$ with

$$\|f * h\|_p \leq \|f\|_p \|h\|_1$$

Proof. By Minkowski's inequality for integrals,

$$\begin{aligned}\left(\int \left|\int f(x-y)h(y)\,dy\right|^p dx\right)^{1/p} &\leq \int \left(\int |f(x-y)|^p\,dx\right)^{1/p} |h(y)|\,dy \\ &= \|f\|_p \int |h(y)|\,dy \\ &= \|f\|_p \|h\|_1 < \infty\end{aligned}$$

Hence, $\int |f(x-y)||h(y)|\,dy < \infty$ a.e. with respect to x and $\|f * h\|_p \leq \|f\|_p \|h\|_1$.

Continuity of Shifts

For f on $\mathbb{R}^d$ and $z \in \mathbb{R}^d$,

$$\tau_z f(x) = f(x + z), \quad x \in \mathbb{R}^d$$

Note that $\|\tau_z f\|_p = \|f\|_p$, $p \in (0, \infty]$. Also note that $\tau_0 f = f$.

Lemma 3

Let $p \in [1, \infty)$, $f \in L^p$. Then,

1. $\|\tau_z f - f\|_p \xrightarrow{z\to 0} 0$ or $\int |f(x-z) - f(x)|^p\,dx \xrightarrow{z\to 0} 0$
2. $\|\tau_z f - \tau_{z_0} f\|_p = \|\tau_{z-z_0} f - f\|_p \xrightarrow{z\to z_0} 0$

The second part follows directly from $\|\tau_z f - f\|_p \xrightarrow{z\to 0} 0$.

Proof. Let $g \in \mathcal{C}_c(\mathbb{R}^d)$. Then, by DCT,

$$\int |g(x+z) - g(x)|^p\,dx \xrightarrow{z\to 0} 0$$

Since if $|z| \leq 1$, $\text{supp}(g) \subseteq B_R(0)$,

$$|g(x+z) - g(x)| \leq |g(x+z)| + |g(x)| \leq \chi_{B_{R+1}}(x) \cdot 2\|g\|_\infty \in L^p$$

Let $f \in L^p$ with $p \in [1, \infty)$. For any $g \in \mathcal{C}_c(\mathbb{R}^d)$,

$$\begin{aligned}
\|\tau_z f - f\|_p &\leq \|\tau_z f - \tau_z g\|_p + \|\tau_z g - g\|_p + \|g - f\|_p \\
&= \|f - g\|_p + \|\tau_z g - g\|_p + \|g - f\|_p \\
&= 2\,\|f - g\|_p + \|\tau_z g - g\|_p
\end{aligned}$$

Since g is continuous,

$$\limsup_{z \to 0} \|\tau_z f - f\|_p \leq 2\,\|f - g\|_p + \limsup_{z \to 0} \|\tau_z g - g\|_p = 2\,\|f - g\|_p$$

Functions in $\mathcal{C}_c(\mathbb{R}^d)$ are dense in L^p. The density property states that for any function $f \in L^p$ and any $\epsilon > 0$, there exists a function $g \in \mathcal{C}_c(\mathbb{R}^d)$ such that $\|f - g\|_p < \epsilon$. Using this density property, for any $\epsilon > 0$, we can find a $g \in \mathcal{C}_c(\mathbb{R}^d)$ with $\|f - g\|_p < \epsilon/2$. This means $2\|f - g\|_p < \epsilon$. Therefore,

$$\limsup_{z \to 0} \|\tau_z f - f\|_p \leq 2\|f - g\|_p < \epsilon$$

Since ϵ is arbitrary,

$$\limsup_{z \to 0} \|\tau_z f - f\|_p = 0$$

Since $\limsup_{z \to 0} \|\tau_z f - f\|_p \geq \liminf_{z \to 0} \|\tau_z f - f\|_p \geq 0$, we conclude $\lim_{z \to 0} \|\tau_z f - f\|_p = 0$.

Mollifiers

Let $\tilde{w}(x) := \ell(|x|_2^2)$, $x \in \mathbb{R}^d$ where $|x| := \sqrt{x_1^2 + \ldots + x_d^2}$. Then,

$$\tilde{w} \in \mathcal{C}_c^\infty(\mathbb{R}^d), \quad \text{supp}(\tilde{w}) = B_1(0)$$

$\mathcal{C}_c^\infty(\mathbb{R}^d)$ is the space of infinitely differentiable functions with compact support. Let $w(x) = c\tilde{w}(x)$, $x \in \mathbb{R}^d$, with $c > 0$ chosen so that $\int w\,dx = 1$. For $\varepsilon > 0$, set

$$w_\varepsilon(x) := \varepsilon^{-d} w\left(\frac{x}{\varepsilon}\right), \quad x \in \mathbb{R}^d$$

Note. $w_\varepsilon \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, $\text{supp}(w_\varepsilon) = B_\varepsilon(0)$ with $w_\varepsilon > 0$, and $\int w_\varepsilon(x)\,dx = 1$ for any $\varepsilon > 0$.

Remark. w is a probability density function of a random variable $\xi \in \mathbb{R}^d$, $|\xi| \leq 1$ almost surely. w_ε is a probability density function of $\varepsilon\xi$.

$$\mathbb{E}[f(\varepsilon\xi)] = \int f(\varepsilon x) w(x)\,dx, \quad y = \varepsilon x \implies \int f(y) w_\varepsilon(y)\,dy$$

Smoothing

Let $p \in [1, \infty)$, $f \in L^p$. Then,

$$f_\varepsilon(x) = (f * w_\varepsilon)(x) = \int f(x-y) w_\varepsilon(y)\, dy = \mathbb{E}[f(x - \varepsilon\xi)], \quad x \in \mathbb{R}^d$$

1. $f_\varepsilon \in L^p$, $\|f_\varepsilon\|_p \le \|f\|_p$, $\|f_\varepsilon - f\|_p \xrightarrow{\varepsilon \to 0} 0$.
2. $f_\varepsilon \in C_b^\infty$, where C_b^∞ is the space of all infinitely differentiable bounded functions with all derivatives bounded.
3. Let $\gamma = (\gamma_1, \ldots, \gamma_d)$, $|\gamma| = \gamma_1 + \ldots + \gamma_d$, and $D^\gamma = \frac{\partial^{|\gamma|}}{\partial x_1^{\gamma_1} \ldots \partial x_d^{\gamma_d}}$. Then,

$$D^\gamma f_\varepsilon(x) = \int D^\gamma w_\varepsilon(x-y) f(y)\, dy = (D^\gamma w_\varepsilon) * f(x), \quad x \in \mathbb{R}^d$$

Proof.

1. By Proposition 2,

$$\|f_\varepsilon\|_p = \|f * w_\varepsilon\|_p \le \|f\|_p \cdot \|w_\varepsilon\|_1 = \|f\|_p$$

Convergence. First, since $\int w_\varepsilon(y)\, dy = 1$,

$$|(f * w_\varepsilon)(x) - f(x)| = \left| \int [f(x-y) - f(x)] w_\varepsilon(y)\, dy \right| \le \int |\tau_{-y} f(x) - f(x)| w_\varepsilon(y)\, dy$$

By Minkowski's inequality for integrals,

$$\|f * w_\varepsilon - f\|_p \le \left\| \int |\tau_{-y} f - f| w_\varepsilon(y)\, dy \right\|_p \le \int \|\tau_{-y} f - f\|_p w_\varepsilon(y)\, dy \le \int \sup_{|y| \le \varepsilon} \|\tau_{-y} f - f\|_p w_\varepsilon(y)\, dy$$

By Lemma 3,

$$\|f * w_\varepsilon - f\|_p \le \int \sup_{|y| \le \varepsilon} \|\tau_{-y} f - f\|_p w_\varepsilon(y)\, dy = \sup_{|y| \le \varepsilon} \|\tau_y f - f\|_p \xrightarrow{\varepsilon \to 0} 0$$

$$\implies \|f_\varepsilon - f\|_p \xrightarrow{\varepsilon \to 0} 0$$

2. Apply Theorem 2.27 and induction.
3. Note that $D^\alpha w_\varepsilon \in C_c^\infty(\mathbb{R}^d) \subset L^q$. $D^\alpha w_\varepsilon$ belongs to L^q because:
   (a) First, note that w_ε is defined as $\varepsilon^{-d} w(x/\varepsilon)$, where $w(x) = C \cdot \ell(|x|/\lambda)$, and ℓ is the standard bump function that is infinitely differentiable and has compact support.
   (b) Since w inherits the properties of ℓ, it is also infinitely differentiable with compact support, meaning $w \in C_c^\infty(\mathbb{R}^d)$.
   (c) When we take any partial derivative D^α of w_ε, the result is still a smooth function with compact support. This is because differentiation preserves smoothness, and the rescaling by ε doesn't change the compact support property - it just scales it.

(d) Any function in $C_c^\infty(\mathbb{R}^d)$ automatically belongs to every L^q space for $q \in [1,\infty]$ because: (1) It's bounded since it's continuous on a compact set. (2) It has finite support (zero outside a compact set). Therefore, the integral $\int |D^\alpha w_\varepsilon(x)|^q\,dx < \infty$ for any $q \in [1,\infty)$ and $\sup_x |D^\alpha w_\varepsilon(x)| < \infty$.

By Proposition 1,

$$\sup_x |D^\alpha f_\varepsilon(x)| = \sup_x |(D^\alpha w_\varepsilon) * f(x)| \le \|f\|_p \|D^\alpha w_\varepsilon\|_q$$

Duals of Continuous Function Spaces

Let X be a compact metric space (CM), and $C(X)$ be the space of all continuous real-valued functions on X with finite norm:

$$|f|_u = \sup_{x\in X} |f(x)| = \max_{x\in X} |f(x)|.$$

We always consider the Borel σ-algebra $\mathcal{B}(X)$ on X. Note that X is compact if any sequence x_n in X has a converging subsequence.

Proposition 1

Let μ be a finite signed measure on X. Define:

$$\ell_\mu(f) = \int f\,d\mu, \quad f \in C(X).$$

Then, $\ell_\mu \in [C(X)]^*$ and

$$|\ell_\mu|_{[C(X)]^*} = \sup_{|f|_{C(X)}\le 1} |\ell_\mu(f)| = \|\ell_\mu\| \le \|\mu\| = |\mu|(X) < \infty,$$

where $|\mu| = \mu^+ + \mu^-$ is the total variation of μ.

Proof. For $f \in C(X)$,

$$|\ell_\mu(f)| = \left|\int f\,d\mu\right| \le \int |f|\,d|\mu| \le |f|_u |\mu|(X).$$

Hence $\ell_\mu \in [C(X)]^*$, $\|\ell_\mu\| \le \|\mu\|$.

Tietze-Urysohn Theorem

Let (E,ρ) be a metric space, $A \subseteq E$ be closed, and $f : A \to \mathbb{R}$ be bounded and continuous. Then, f has a continuous extension $g : E \to \mathbb{R}$ such that:

$$\sup_{x\in E} g(x) = \sup_{x\in A} f(x), \quad \inf_{x\in E} g(x) = \inf_{x\in A} f(x),$$

$$g(x) = \begin{cases} f(x), & x \in A \\ \inf_{y\in A} \frac{f(y)\rho(x,y)}{\rho(x,A)}, & x \notin A \end{cases},$$

where $\rho(x,A) = \inf_{y\in A} \rho(x,y)$. The ratio $\frac{\rho(x,A)}{\rho(x,y)} \ge 1$. Here is the reason: For a fixed x,

- When $y \in A$ is close to x, this ratio approaches 1.
- When $y \in A$ is far from x, this ratio is smaller than 1.

$$\inf_{y \in A} f(y) \leq \inf_{y \in A} \frac{f(y)\rho(x,y)}{\rho(x,A)}$$

.

Corollary 0

Let $A, B \subseteq E$ be closed and disjoint subsets of a metric space (E, ρ). Then, there exists a continuous $g : E \to [0,1]$ such that

$$g|_A = 1, \quad g|_B = 0.$$

Proof. Applying Tietze-Urysohn Theorem to the closed set $A \cup B$,

$$f(x) = \begin{cases} 1, & x \in A \\ 0, & x \in B \end{cases}.$$

Lemma 1

Let X be a compact metric space, $K \subseteq U \subseteq X$ where K is compact and U is open. Then, there exists open $V \subseteq X$ such that $K \subseteq V \subseteq \overline{V} \subseteq U$ with $\overline{V}$ compact.

Proof. For all $x \in K$, there exists an open ball V_x and closed ball N_x, both centered at x, such that

$$V_x \subseteq N_x \subseteq U, \quad K \subseteq \bigcup_{x \in K} V_x.$$

Since K is compact, every open cover of K has a finite subcover. Then, there exists $x_1, \dots, x_m \in K$ so that

$$K \subseteq \bigcup_{j=1}^{m} V_{x_j} \subseteq \bigcup_{j=1}^{m} N_{x_j} \subseteq U.$$

Then, we set

$$V = \bigcup_{j=1}^{m} V_{x_j}.$$

The $\overline{V_{x_j}}$ is the smallest closed set containing V_{x_j}. Since N_{x_j} is closed and $V_{x_j} \subseteq N_{x_j}$, it follows that:

$$\overline{V_{x_j}} \subseteq N_{x_j}.$$

In general, the closure of a union satisfies $\overline{A \cup B} = \overline{A} \cup \overline{B}$. Therefore, for $V = \bigcup_{j=1}^{m} V_{x_j}$,

$$\overline{V} = \overline{\bigcup_{j=1}^{m} V_{x_j}} = \bigcup_{j=1}^{m} \overline{V_{x_j}},$$

Since each $\overline{V_{x_j}} \subseteq N_{x_j}$,

$$\overline{V} \subseteq \bigcup_{j=1}^{m} N_{x_j} \implies V \subseteq \overline{V} \subseteq \bigcup_{j=1}^{m} N_{x_j} \subseteq U.$$

$\bigcup_{j=1}^{m} N_{x_j}$ is compact because each closed ball N_{x_j} is bounded and closed. The finite union of compact sets is compact. Since $\overline{V}$ is a closed subset of the compact set $\bigcup_{j=1}^{m} N_{x_j}$, it must also be compact.

Notation. Let $U \subseteq X$ be open, $f \in C(X)$.

$$f \prec U \text{ if } 0 \leq f \leq 1 \text{ and } \operatorname{supp}(f) \subseteq U.$$

Proposition 1

Let X be a compact metric space, $K \subseteq U \subseteq X$ where K is compact and U is open. Then, there exists $f \prec U$ such that $f|_K = 1$.

Proof. By Lemma 1, there exists open V such that

$$K \subseteq V \subseteq \overline{V} \subseteq U$$

By Corollary 0, there exists $f \in C(X)$ such that

$$f|_K = 1, \quad f|_{V^c} = 0$$

Note $\operatorname{supp}(f) \subseteq \overline{V} \subseteq U$. By Corollary 0, $0 \leq f \leq 1$ ($g : E \to [0, 1]$).

Corollary 1

Let X be a compact metric space, $K \subseteq \bigcup_{j=1}^{n} U_j$ where K is compact and U_j are open for all $j = 1, \ldots, n$. Then, there exists $f_j \prec U_j$, $j = 1, \ldots, n$, such that

$$\left.\left(\sum_{j=1}^{n} f_j\right)\right|_K = 1$$

Proof. For any $x \in K$, there exists an open ball V_x and closed ball N_x, both centered at x such that:

$$V_x \subseteq N_x \subseteq U_j \text{ for some } j \in \{1, \ldots, n\}$$

Hence, $K \subseteq \bigcup_{x \in K} V_x$ and there exists $x_1, \ldots, x_m$ so that

$$K \subseteq \bigcup_{k=1}^{m} V_{x_k} \subseteq \bigcup_{k=1}^{m} N_{x_k}$$

Let K_j be the union of all N_{x_k} s.t. $N_{x_k} \subseteq U_j$. Then $K \subseteq \bigcup_{j=1}^{n} K_j$ and all K_j are compact with $K_j \subseteq U_j$.

By Proposition 1, there exists $g_j \prec U_j, g_j|_{K_j} = 1$. Let $U = \bigcup_{j=1}^n \{x : g_j(x) > 0\}$. Then, $\bigcup_{j=1}^n K_j \subseteq U$ and open since $g_j \in C(X)$. Specifically, the set $\{x : g_j(x) > 0\}$ is open as the preimage of the open set $(0, \infty)$ under the continuous function g_j.

By Proposition 1, there exists $f \prec U$ such that $f|_{\bigcup_{j=1}^n K_j} = 1$. Set:

$$g_{n+1} = 1 - f = \begin{cases} 0 & \text{on } \bigcup_{j=1}^n K_j \\ 1 & \text{on } U^c \end{cases}$$

On K, since $K \subseteq \bigcup_{j=1}^n K_j$, for any $x \in K$, at least one $g_j(x) = 1$, so $f(x) \geq 1$. On U^c, all $g_j = 0$, so $f = 0$. Therefore, on $\bigcup_{j=1}^n K_j$, $f = 1$, so $g_{n+1} = 0$.

$$f_j = \frac{g_j}{\sum_{i=1}^{n+1} g_i}, \quad j = 1, \ldots, n$$

Since $g_{n+1} \geq 0$, we must have $g_{n+1} = 0$ on $\bigcup_{j=1}^n K_j$. On U^c, $f = 0$, so $g_{n+1} = 1$.

Since $g_{n+1} = 0$ on K,

$$\sum_{i=1}^{n+1} g_i = \sum_{j=1}^{n} g_j \text{ on } K \implies f_j = \frac{g_j}{\sum_{k=1}^n g_k} \text{ on } K \implies \sum_{j=1}^n f_j = \frac{\sum_{j=1}^n g_j}{\sum_{k=1}^n g_k} = 1 \text{ on } K$$

$$\implies \left(\sum_{j=1}^n f_j \right) \Bigg|_K = 1$$

Since $g_j \prec U_j$, $\mathrm{supp}(g_j) \subseteq U_j$. The denominator $\sum_{i=1}^{n+1} g_i$ is positive everywhere, so dividing by it does not change the support. Thus, $\mathrm{supp}(f_j) = \mathrm{supp}(g_j) \subseteq U_j$, meaning $f_j \prec U_j$.

Representations of Positive Linear Functionals

A linear functional $\ell : C(X) \to \mathbb{R}$ is **positive** if $\ell(f) \geq 0$ if $f \geq 0$.

Remark. If $f \geq g$, then $\ell(f) \geq \ell(g)$ for positive ℓ. In fact, ℓ is positive iff this holds.

Proposition 2

If ℓ is positive, then $\ell \in [C(X)]^*$.

Proof. Let $f \in C(X)$. Then,

$$-|f|_u \leq f \leq |f|_u \implies -|f|_u \ell(1) \leq \ell(f) \leq |f|_u \ell(1) \implies |\ell(f)| \leq \ell(1)|f|_u.$$

Definition

Let μ be a Borel measure on a metric space X. Let $A \in \mathcal{B}$ (Borel σ-algebra). We say μ is:

(i) **Outer regular** on A if

$$\mu(A) = \inf\{\mu(U) : U \supseteq A, U \text{ is open}\}$$

(ii) **Inner regular** on A if

$$\mu(A) = \sup\{\mu(K) : K \subseteq A, K \text{ is compact}\}$$

(iii) **Regular** if μ is both inner and outer regular for all $E \in \mathcal{B}$.

Theorem 1

Let $\ell : C(X) \to \mathbb{R}$ be a positive linear functional. Then, there exists a unique finite Borel regular measure μ on X such that:

$$\ell(f) = \int f d\mu, \quad f \in C(X)$$

(i) For open $U \subseteq X$,

$$\mu(U) = \sup\{\ell(f) : f \prec U\} \leq \ell(1) = \mu(X)$$
$$\ell(1) = \int_X 1\, d\mu = \mu(X).$$

(ii) For compact $K \subseteq X$,

$$\mu(K) = \inf\{\ell(f) : f \geq \chi_K, f \in C(X)\}.$$

Note. For open $U \subseteq X$,

$$\chi_U(x) = \sup\{f(x) : f \prec U\}, \quad x \in X.$$

Comment. (i) implies (ii): if K is compact,

$$\begin{aligned}
\mu(K) &= \mu(X) - \mu(K^c) \\
&= \ell(1) - \sup\{\ell(f) : f \prec K^c\} \\
&= \ell(1) - \sup\{\ell(f) : 0 \leq f \leq 1, \operatorname{supp}(f) \subseteq K^c\} \\
&= \inf\{\ell(1-f) : 0 \leq f \leq 1, \operatorname{supp}(f) \subseteq K^c\} \\
&\geq \inf\{\ell(g) : g \geq \chi_K, g \in C(X)\}
\end{aligned}$$

For f with $\operatorname{supp}(f) \subseteq K^c$, f is zero on K since K and $\operatorname{supp}(f)$ are disjoint. Thus, $1 - f \geq 1$ on K because $f = 0$ on K, and $1 - f \geq 0$ everywhere since $f \leq 1$. Therefore, $1 - f \geq \chi_K$, where χ_K is the indicator function of K.

Proof. *Uniqueness.* Let μ be a finite regular measure and $\ell(f) = \int f d\mu$. Let $U \subseteq X$ be open. By the note above,

$$\mu(U) = \int_U 1\, d\mu = \sup\{\ell(f) : f \prec U\}$$

Let $K \subseteq U$ be compact. By Proposition 1, there exists $f \prec U$ such that $f|_K = 1$. Then,

$$\mu(K) \leq \int_X f d\mu = \ell(f)$$

$$\implies \mu(U) = \sup\{\mu(K') : K' \subseteq U, K' \text{ compact}\} \leq \sup\{\ell(f) : f \prec U\}$$
$$\implies \mu(U) = \sup\{\ell(f) : f \prec U\} \text{ which is (i)}$$

Why does it imply uniqueness? Suppose there were two measures μ_1 and μ_2 representing ℓ:

$$\ell(f) = \int f \, d\mu_1 = \int f \, d\mu_2 \quad \forall f \in C(X).$$

For any open set U, both $\mu_1(U)$ and $\mu_2(U)$ must satisfy:

$$\mu_i(U) = \sup\{\ell(f) : f \prec U\}, \quad i = 1, 2.$$

Since the right-hand side depends only on ℓ, we must have:

$$\mu_1(U) = \mu_2(U).$$

Thus, μ_1 and μ_2 agree on all open sets. The measure μ is outer regular, meaning for any Borel set $E \subseteq X$,

$$\mu(E) = \inf\{\mu(U) : U \supseteq E, U \text{ open}\}.$$

Since μ_1 and μ_2 agree on open sets, they must assign the same measure to any Borel set E:

$$\mu_1(E) = \inf\{\mu_1(U) : U \supseteq E\} = \inf\{\mu_2(U) : U \supseteq E\} = \mu_2(E).$$

Hence, $\mu_1 = \mu_2$ on the entire Borel σ-algebra. In short, open sets generate the topology of X, and their measures are uniquely specified by ℓ. Since μ is outer regular, its values on arbitrary sets are determined by its values on open sets.

Existence.

$$\mu(U) := \sup\{\ell(f) : f \prec U\} \text{ for } U \subseteq X \text{ open}.$$
$$\mu^*(E) := \inf\{\mu(U) : U \supseteq E, U \text{ open}\} \text{ for } E \subseteq X.$$

Claim 1. μ^* is an outer measure.

An outer measure must satisfy:

1. $\mu^*(\emptyset) = 0$.
2. $A \subseteq B \implies \mu^*(A) \leq \mu^*(B)$.
3. $\mu^*\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mu^*(E_n)$.

$\mu^*(\emptyset) = 0$ because $\emptyset$ is covered by itself, and $\mu(\emptyset) = 0$. If $A \subseteq B$, any open cover of B also covers A, so the infimum for A is over a larger set of covers, so $\mu^*(A) \leq \mu^*(B)$. Thus, it is enough to prove that

$$\mu^*\left(\bigcup_{n=1}^{\infty} U_n\right) \leq \sum_{n=1}^{\infty} \mu^*(U_n) \text{ for } U_n \text{ open.}$$

The key observation is that open sets can be used to approximate arbitrary sets when defining the outer measure. There exists $f \prec \bigcup_{n=1}^{\infty} U_n = U$ such that $\ell(f) > \mu\left(\bigcup_{n=1}^{\infty} U_n\right) - \epsilon$ for a given $\epsilon > 0$ since $\mu(U) = \sup\{\ell(f) : f \prec U\}$. Since every open cover of a compact set has a finite subcover, there exists N such that $\text{supp}(f) \subseteq \bigcup_{n=1}^{N} U_n$. By Corollary 1, there exists $f_n \prec U_n$ such that

$$\left(\sum_{n=1}^{N} f_n\right)\Bigg|_{\text{supp}(f)} = 1, \quad f = \sum_{n=1}^{N} (f_n f)$$

$$\mu\left(\bigcup_{n=1}^{\infty} U_n\right) \leq \ell(f) + \epsilon = \left[\sum_{n=1}^{N} \ell(f f_n)\right] + \epsilon \leq \left[\sum_{n=1}^{N} \mu(U_n)\right] + \epsilon$$

Hence μ^* is an outer measure.

Claim 2. μ has a finite regular Borel extension.

By Caratheodory's theorem, μ^* is a measure on the σ-algebra $\mathcal{F}$ of all $A \subseteq X$ such that

$$\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \setminus A) \text{ for all } E \subseteq X \tag{2}$$

We now show $\mathcal{F}$ contains all the open sets of X, i.e., $U \in \mathcal{F}$ if $U \subseteq X$ is open. If this is the case, $\mathcal{F}$ contains $\mathcal{B}$ and μ can be extended to a measure on $\mathcal{B}$.

Let $U \subseteq X$ be open. Equation 2 holds if E is open. In this case, $E \cap U$ is open so there exists $f \prec E \cap U$ such that $\ell(f) > \mu(E \cap U) - \frac{\epsilon}{2}$ where $\epsilon > 0$. There exists $g \prec E \setminus \text{supp}(f)$ such that

$$\ell(g) > \mu(E \setminus \text{supp}(f)) - \frac{\epsilon}{2}$$

Then, $f + g \prec E$ since $\text{supp}(f), \text{supp}(g) \subseteq E$. Then,

$$\begin{aligned} \mu(E) \geq \ell(f+g) &= \ell(f) + \ell(g) \\ &> \mu(E \cap U) - \frac{\epsilon}{2} + \mu(E \setminus \text{supp}(f)) - \frac{\epsilon}{2} \\ &\geq \mu(E \cap U) + \mu^*(E \setminus U) - \epsilon \end{aligned}$$

Now, we prove Equation 2 holds for arbitrary $E \subseteq X$. For all $\epsilon > 0$, there exists open $\tilde{E} \supset E$ such that

$$\mu^*(E) \geq \mu(\tilde{E}) - \epsilon$$
$$\implies \mu^*(E) \geq \mu(\tilde{E}) - \epsilon \geq \mu^*(\tilde{E} \cap U) + \mu^*(\tilde{E} \setminus U) - \epsilon \geq \mu^*(E \cap U) + \mu^*(E \setminus U) - \epsilon$$

Since $\mu(U) \leq \ell(1) < \infty$ for all open $U \subseteq X$, μ is finite.

We define for any subset $E \subseteq X$ by:

$$\mu^*(E) = \inf\{\mu(U) : U \supseteq E,\ U \text{ open}\}.$$

This ensures μ^* is outer regular. Why Outer Regularity on the Borel σ-Algebra Implies Inner Regularity (in a Compact Space)? Let $A \in \mathcal{B}$ be a Borel set. Since μ is outer regular, for the complement A^c, there exists an open set $U \supseteq A^c$ such that:

$$\mu(U) \leq \mu(A^c) + \epsilon \text{ for any } \epsilon > 0.$$

Since $U \supseteq A^c$, its complement $K = U^c$ satisfies $K \subseteq A$. In a compact space, closed subsets are compact, so K is compact. From $\mu(X) = \mu(A) + \mu(A^c)$ and $\mu(U) \leq \mu(A^c) + \epsilon$,

$$\mu(K) = \mu(U^c) = \mu(X) - \mu(U) \geq \mu(X) - (\mu(A^c) + \epsilon) = \mu(A) - \epsilon.$$

Since $K \subseteq A$, this shows:

$$\mu(A) \leq \mu(K) + \epsilon \leq \sup\{\mu(K') : K' \subseteq A,\ K' \text{ compact}\} + \epsilon.$$

Taking $\epsilon \to 0$ gives:

$$\mu(A) \leq \sup\{\mu(K) : K \subseteq A,\ K \text{ compact}\}.$$

The reverse inequality ($\geq$) is trivial since $\mu(K) \leq \mu(A)$ for $K \subseteq A$. Thus,

$$\mu(A) = \sup\{\mu(K) : K \subseteq A,\ K \text{ compact}\}.$$

We conclude μ is inner regular. Since μ is both outer and inner regular, μ is regular.

Claim 3. $\int f d\mu = \ell(f), f \in C(X)$.

Let $0 \leq f \leq 1, f \in C(X)$. Let $N > 1$. Then,

$$\begin{aligned} f &= \sum_{n=1}^{N} f\chi_{\frac{n-1}{N} \leq f < \frac{n}{N}} \\ &= \sum_{n=1}^{N} \left[\left(f - \frac{n-1}{N} \right) \chi_{\frac{n-1}{N} \leq f < \frac{n}{N}} + \frac{n-1}{N} \chi_{\frac{n-1}{N} \leq f < \frac{n}{N}} \right] \end{aligned}$$

$$\implies f = \sum_{n=1}^{N} f\chi_{\frac{n-1}{N} \leq f < \frac{n}{N}} = \sum_{n=1}^{N} \left[\left(f - \frac{n-1}{N} \right) \chi_{\frac{n-1}{N} \leq f \leq \frac{n}{N}} + \frac{1}{N} \chi_{\frac{n}{N} \leq f} \right] = \sum_{n=1}^{N} f_n$$

where $f_n \in C(X)$.

$$\begin{aligned} &\sum_{n=1}^{N} \frac{n-1}{N} \left(\chi_{\frac{n-1}{N} \leq f} - \chi_{\frac{n}{N} \leq f} \right) \\ &= \frac{1}{N} \sum_{n=1}^{N} (n-1) \left(\chi_{\frac{n-1}{N} \leq f} - \chi_{\frac{n}{N} \leq f} \right) \\ &= \frac{1}{N} \left[1 \cdot \left(\chi_{\frac{1}{N} \leq f} - \chi_{\frac{2}{N} \leq f} \right) + 2 \cdot \left(\chi_{\frac{2}{N} \leq f} - \chi_{\frac{3}{N} \leq f} \right) + \ldots + (N-1) \cdot \left(\chi_{\frac{N-1}{N} \leq f} - \chi_{\frac{N}{N} \leq f} \right) \right] \\ &= \frac{1}{N} \left[\chi_{\frac{1}{N} \leq f} - \chi_{\frac{2}{N} \leq f} + 2\chi_{\frac{2}{N} \leq f} - 2\chi_{\frac{3}{N} \leq f} + \ldots - (N-1)\chi_{\frac{N}{N} \leq f} \right] \\ &= \frac{1}{N} \left[\chi_{\frac{1}{N} \leq f} + \chi_{\frac{2}{N} \leq f} + \chi_{\frac{3}{N} \leq f} + \ldots + \chi_{\frac{N-1}{N} \leq f} - (N-1) \cdot \chi_{\frac{N}{N} \leq f} \right] \end{aligned}$$

Since $(N-1) \cdot \chi_{\frac{N}{N} \leq f} = 0$,

$$\sum_{n=1}^{N} \frac{n-1}{N} \chi_{\frac{n-1}{N} \leq f < \frac{n}{N}} = \sum_{n=1}^{N} \frac{n-1}{N} \left(\chi_{\frac{n-1}{N} \leq f} - \chi_{\frac{n}{N} \leq f} \right) = \frac{1}{N} \sum_{n=1}^{N} \chi_{\frac{n}{N} \leq f}$$

For each n,

$$f_n = \left(f - \frac{n-1}{N} \right) \chi_{\frac{n-1}{N} \leq f \leq \frac{n}{N}} + \frac{1}{N} \chi_{\frac{n}{N} \leq f} \implies \frac{1}{N} \chi_{\frac{n}{N} \leq f} \leq f_n \leq \frac{1}{N} \chi_{\frac{n-1}{N} \leq f}$$

By (ii),

$$\mu \left(\frac{n}{N} \leq f \right) \leq N \ell(f_n) \leq \mu \left(\frac{n-1}{N} \leq f \right)$$

$$\implies \frac{1}{N} \sum_{n=1}^{N} \mu \left(\frac{n}{N} \leq f \right) \leq \sum_{n=1}^{N} \ell(f_n) \leq \frac{1}{N} \sum_{n=1}^{N} \mu \left(\frac{n-1}{N} \leq f \right)$$

Focus on the right-hand side (RHS):

$$\ell(f) \leq \sum_{n=1}^{N} \frac{1}{N} \mu \left(\frac{n-1}{N} \leq f \right)$$

We can express $\mu \left(\frac{n-1}{N} \leq f \right)$ as:

$$\mu \left(\frac{n-1}{N} \leq f \right) = \sum_{k=n}^{N} \mu \left(\frac{k-1}{N} \leq f < \frac{k}{N} \right) + \mu(f \geq 1)$$

Since $f \leq 1$,

$$\mu \left(\frac{n-1}{N} \leq f \right) = \sum_{k=n}^{N} \mu \left(\frac{k-1}{N} \leq f < \frac{k}{N} \right)$$

Substitute back into the RHS:

$$\ell(f) \leq \sum_{n=1}^{N} \frac{1}{N} \sum_{k=n}^{N} \mu \left(\frac{k-1}{N} \leq f < \frac{k}{N} \right)$$

Interchange the order of summation:

$$\sum_{n=1}^{N} \sum_{k=n}^{N} \frac{1}{N} \mu \left(\frac{k-1}{N} \leq f < \frac{k}{N} \right) = \sum_{k=1}^{N} \sum_{n=1}^{k} \frac{1}{N} \mu \left(\frac{k-1}{N} \leq f < \frac{k}{N} \right)$$

$$\implies \sum_{n=1}^{k} \frac{1}{N} \mu \left(\frac{k-1}{N} \leq f < \frac{k}{N} \right) = \frac{k}{N} \mu \left(\frac{k-1}{N} \leq f < \frac{k}{N} \right)$$

$$\implies \ell(f) \leq \sum_{k=1}^{N} \frac{k}{N} \mu \left(\frac{k-1}{N} \leq f < \frac{k}{N} \right)$$

Write $\frac{k}{N} = \frac{k-1}{N} + \frac{1}{N}$:

$$\ell(f) \leq \sum_{k=1}^{N} \left(\frac{k-1}{N} + \frac{1}{N} \right) \mu \left(\frac{k-1}{N} \leq f < \frac{k}{N} \right)$$

$$\implies \ell(f) \leq \sum_{k=1}^{N} \frac{k-1}{N} \mu \left(\frac{k-1}{N} \leq f < \frac{k}{N} \right) + \frac{1}{N} \sum_{k=1}^{N} \mu \left(\frac{k-1}{N} \leq f < \frac{k}{N} \right)$$

The second sum is:

$$\frac{1}{N} \sum_{k=1}^{N} \mu \left(\frac{k-1}{N} \leq f < \frac{k}{N} \right) = \frac{1}{N} \mu(f > 0)$$

because the sets $\frac{k-1}{N} \leq f < \frac{k}{N}$ partition $\{f > 0\}$. Then,

$$\sum_{n=1}^{N} \frac{n-1}{N} \mu \left(\frac{n-1}{N} \leq f < \frac{n}{N} \right) \leq \ell(f) \leq \sum_{n=1}^{N} \frac{n-1}{N} \mu \left(\frac{n-1}{N} \leq f < \frac{n}{N} \right) + \frac{1}{N} \mu(f > 0)$$

The left-hand side is the lower Riemann sum and the right-hand side is the upper Riemann sum. As we integrate (Riemann sum) and let $N \to \infty$,

$$\int f d\mu = \ell(f) = \int f d\mu$$

Therefore, $\int f d\mu = \ell(f)$ for all $f \in C(X)$.

Corollary 2

Let X be a compact metric space. Then, any finite Borel measure γ is regular.

Proof. Let $\ell(f) = \int f \, d\gamma$, $f \in C(X)$. Then, ℓ is positive. By Theorem 1, there exists a regular finite μ so that:

$$\ell(f) = \int f \, d\mu = \int f \, d\gamma \text{ for any } f \in C(X).$$

Let $K \subset X$ be compact and define:

$$F_n = \{x \in X : \rho(x, K) \geq \frac{1}{n}\} \text{ for all } n \geq 1.$$

We show that the complement $F_n^c = \{x \in X : \rho(x, K) < 1/n\}$ is open: Take any $x \in F_n^c$, meaning $\rho(x, K) = d < 1/n$. By definition of infimum, there exists $y \in K$ such that $\rho(x, y) < (1/n + d)/2$. Now, consider the open ball $B(x, r)$ where $r = (1/n - d)/2 > 0$. For any $z \in B(x, r)$, by the triangle inequality:

$$\rho(z, K) \leq \rho(z, y) \leq \rho(z, x) + \rho(x, y) < r + \frac{1/n + d}{2} = \frac{1/n - d}{2} + \frac{1/n + d}{2} = 1/n.$$

Thus, $\rho(z, K) < 1/n$, so $z \in F_n^c$. Since every $x \in F_n^c$ has an open neighborhood entirely contained in F_n^c, F_n^c is open, and hence F_n is closed.

By Corollary 0, there exists continuous $f_n : X \to [0, 1]$ such that $f_n|_K = 1$, $f_n|_{F_n} = 0$. Note that $f_n(x) \to \chi_K(x)$ for all $x \in X$ and:

$$\int f_n \, d\mu = \int f_n \, d\gamma \implies \mu(K) = \gamma(K)$$

Therefore, γ is regular, and in fact $\gamma = \mu$, since μ is regular. For open U,

$$\gamma(U) = \gamma(X) - \gamma(U^c) = \mu(X) - \mu(U^c) = \mu(U)$$

Let $A \in \mathcal{B}$, $\epsilon > 0$. Since μ is regular, there exists $U_\epsilon \supseteq A \supseteq K_\epsilon$ so that:

$$\mu(U_\epsilon \setminus K_\epsilon) < \epsilon \implies \gamma(U_\epsilon) \leq \gamma(K_\epsilon) + \epsilon \leq \gamma(A) + \epsilon$$

Hence, γ is outer regular on A. Similarly,

$$\mu(K_\epsilon) \geq \mu(A) - \epsilon \implies \gamma(K_\epsilon) \geq \gamma(A) - \epsilon.$$

Thus, γ is inner regular. Then, γ is regular and $\gamma = \mu$ since γ and μ are equal on compact and open sets.

Lemma 2

Let $\ell \in [C(X)]^*$. Then, $\ell = \ell_1 - \ell_2$ with ℓ_1, ℓ_2 positive.

Proof. Define $P = \{f \in C(X) : f \geq 0\}$. For any $f \in P$,

$$\ell_1(f) := \sup\{\ell(g) : 0 \leq g \leq f, g \in C(X)\}$$

Note that $\ell_1(f) \geq 0$ since we can take $g = 0$ in the supremum. We first show that ℓ_1 is linear on P. For any $f, f' \in P$ and $c \geq 0$, we need to verify:

1. *Additivity.* $\ell_1(f + f') = \ell_1(f) + \ell_1(f')$.

   For any $0 \leq g \leq f$ and $0 \leq g' \leq f'$, we have $0 \leq g + g' \leq f + f'$, and thus:

   $$\ell(g) + \ell(g') = \ell(g + g') \leq \ell_1(f + f')$$

   Taking suprema over g and g' gives $\ell_1(f) + \ell_1(f') \leq \ell_1(f + f')$. For the reverse inequality, given any $0 \leq h \leq f + f'$, we can write $h = u + u'$ where $0 \leq u \leq f$ and $0 \leq u' \leq f'$ by taking $u = \min(h, f)$ and $u' = h - u$. Then,

   $$\ell(h) = \ell(u) + \ell(u') \leq \ell_1(f) + \ell_1(f')$$

   Taking supremum over h gives $\ell_1(f + f') \leq \ell_1(f) + \ell_1(f')$.

2. *Homogeneity.* $\ell_1(cf) = c\ell_1(f)$.

$$\ell_1(cf) = \sup\{\ell(g) : 0 \leq g \leq cf\} = \sup\{\ell(ch) : 0 \leq h \leq f\} = c \sup\{\ell(h) : 0 \leq h \leq f\}$$

$$\implies \ell_1(cf) = c\ell_1(f)$$

We now extend ℓ_1 to all of $C(X)$. Any $f \in C(X)$ can be written as $f = f_1 - f_2$ where $f_1, f_2 \in P$, using positive and negative parts. Define:

$$\ell_1(f) := \ell_1(f_1) - \ell_1(f_2)$$

If $f = f_1 - f_2 = f_1' - f_2'$, then $f_1 + f_2' = f_1' + f_2$, so by additivity on P:

$$\ell_1(f_1) + \ell_1(f_2') = \ell_1(f_1') + \ell_1(f_2)$$

$$\implies \ell_1(f_1) - \ell_1(f_2) = \ell_1(f_1') - \ell_1(f_2')$$

Thus, $\ell_1(f) = \ell_1(f_1) - \ell_1(f_2)$ is well-defined. Finally, define $\ell_2 := \ell_1 - \ell$. To see that ℓ_2 is positive, note that for any $f \geq 0$:

$$\ell_2(f) = \ell_1(f) - \ell(f) = \sup\{\ell(g) : 0 \leq g \leq f\} - \ell(f) \geq \ell(f) - \ell(f) = 0$$

Thus, ℓ_1 and ℓ_2 are both positive linear functionals with $\ell = \ell_1 - \ell_2$.

Proposition 4

Let X be a compact metric space, μ be a finite regular Borel measure, $p \in [1, \infty)$. Then, $C(X)$ is dense in L^p.

Proof. $\Sigma = \mathcal{E}$ as $\mu(X) < \infty$. Since $\Sigma = \mathcal{E}$ is dense in L^p, it is enough to approximate χ_E in L^p by continuous functions. Let $E \in \mathcal{B}$, $\epsilon > 0$. Since μ is regular, for any Borel set E and $\epsilon > 0$, there exist a compact set $K_\epsilon \subseteq E$ and an open set $U_\epsilon \supseteq E$, such that:

$$\mu(U_\epsilon) < \mu(E) + \frac{\epsilon}{2}, \quad \mu(K_\epsilon) > \mu(E) - \frac{\epsilon}{2}.$$

Subtracting these inequalities gives:

$$\mu(U_\epsilon \setminus K_\epsilon) = \mu(U_\epsilon) - \mu(K_\epsilon) < \epsilon.$$

Since $K_\epsilon \subseteq E \subseteq U_\epsilon$, by Corollary 0, we can construct a continuous function f such that: $f = 1$ on K_ϵ, $f = 0$ outside U_ϵ, and $0 \leq f \leq 1$ everywhere. The difference $|\chi_E - f|$ is nonzero only on $U_\epsilon \setminus K_\epsilon$, because:

- On K_ϵ, $\chi_E = 1$ and $f = 1$, so $|\chi_E - f| = 0$.
- Outside U_ϵ, $\chi_E = 0$ and $f = 0$, so $|\chi_E - f| = 0$.
- On $U_\epsilon \setminus K_\epsilon$, $|\chi_E - f| \leq 1$ since $0 \leq f \leq 1$ and $\chi_E \in \{0, 1\}$.

Therefore:

$$\int_X |\chi_E - f|^p \, d\mu \leq \int_{U_\epsilon \setminus K_\epsilon} 1 \, d\mu = \mu(U_\epsilon \setminus K_\epsilon) < \epsilon.$$

Since simple functions are dense in L^p, and we can approximate any simple function by continuous functions, we can approximate any L^p-function by continuous functions. Thus, $C(X)$ is dense in $L^p(X, \mu)$.

Lusin's theorem

Let X be a compact metric space and μ be finite. Let f be a bounded measurable function on X ($|f|_\infty \leq C$). Then, for all $\epsilon > 0$, there exists $g \in C(X)$ such that $\mu(\{f \neq g\}) \leq \epsilon$. Moreover, $|g|_\infty \leq \sup_x |f(x)|$, and $0 \leq g \leq \sup_x |f(x)|$ if $f \geq 0$.

Proof. By Proposition 4, since $C(X)$ is dense in $L^1(X,\mu)$, there exists a sequence $\{f_n\} \subset C(X)$ such that $f_n \to f$ in L^1, and hence $f_n \to f$ almost everywhere. Since $f_n \to f$ almost everywhere and $\mu(X) < \infty$, Egoroff's Theorem guarantees that for any $\epsilon > 0$, there exists a measurable set $A \subset X$ such that:

$$\mu(X \setminus A) \leq \frac{\epsilon}{2},$$

and $f_n \to f$ uniformly on A. Since μ is regular from Corollary 2, for the set A, there exists a compact subset $K \subseteq A$ such that:

$$\mu(A \setminus K) \leq \frac{\epsilon}{2}.$$

$$\implies \mu(X \setminus K) = \mu(X \setminus A) + \mu(A \setminus K) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since $f_n \to f$ uniformly on A, and $K \subseteq A$, the restriction $f|_K$ is continuous as a uniform limit of continuous functions. By the Tietze-Urysohn Theorem, since K is closed in X and X is a compact metric space, there exists a continuous function $g \in C(X)$ such that:

$$g|_K = f|_K, \quad \sup_X g = \sup_K f, \quad \inf_X g = \inf_K f.$$

By construction, $f = g$ on K and $\mu(X \setminus K) \leq \epsilon$. Therefore:

$$\{x \in X : f(x) \neq g(x)\} \subseteq X \setminus K, \implies \mu(\{f \neq g\}) \leq \mu(X \setminus K) \leq \epsilon.$$

Since g is an extension of $f|_K$, and f is bounded by C,

$$|g(x)| \leq \sup_K |f| \leq \sup_X |f| = C.$$

If $f \geq 0$, then $g \geq 0$ since $\inf_X g \geq \inf_K f \geq 0$.

Proposition 5

Let X be a compact metric space, μ be a finite signed measure. Let $\ell_\mu(f) = \int f d\mu$, $f \in C(X)$. Then, $\ell_\mu \in [C(X)]^*$ and $\|\ell_\mu\| = |\ell_\mu|_{[C(X)]^*} = |\mu|(X) = \|\mu\|$.

Proof. By Proposition 1, since μ is a finite signed measure, $\ell_\mu \in [C(X)]^*$ and $\|\ell_\mu\| \leq \|\mu\|$. Let D^+, D^- be the disjoint sets of Hahn's decomposition such that $d|\mu| = \chi_{D^+} d\mu - \chi_{D^-} d\mu$ and $d\mu = (\chi_{D^+} - \chi_{D^-}) d|\mu|$. Note that $|\rho| = \rho^2 = 1$ where $\rho = \chi_{D^+} - \chi_{D^-}$. Let $\epsilon > 0$. By

Lusin's theorem, there exists $g \in C(X)$ such that $|\mu|(\{\rho \neq g\}) \leq \epsilon$ and $|g|_\infty \leq 1$. Then:

$$\begin{aligned}
\ell_\mu(g) &= \int g\, d\mu \\
&= \int g\rho\, d|\mu| \\
&= \int_{g=\rho} g\rho\, d|\mu| + \int_{g\neq\rho} g\rho\, d|\mu| \\
&= \int_X \rho^2\, d|\mu| - \int_{g\neq\rho} \rho^2\, d|\mu| + \int_{g\neq\rho} g\rho\, d|\mu| \\
&\geq |\mu|(X) - \epsilon - \epsilon \\
&= |\mu|(X) - 2\epsilon
\end{aligned}$$

Since:

$$0 \leq \int_{g\neq\rho} \rho^2\, d|\mu| = |\mu|(\{g \neq \rho\}) \leq \epsilon$$

$$\left|\int_{g\neq\rho} g\rho\, d|\mu|\right| \leq \int_{g\neq\rho} |g\rho|\, d|\mu| \leq \int_{g\neq\rho} d|\mu| = |\mu|(\{g \neq \rho\}) \leq \epsilon \implies \int_{g\neq\rho} g\rho\, d|\mu| \geq -\epsilon$$

Therefore,

$$\|\mu\| \geq |\ell_\mu(g)| \geq \ell_\mu(g) \geq |\mu|(X) - 2\epsilon$$

Taking $\epsilon \to 0$, we have $\|\ell_\mu\| = \|\mu\|$.

Notation. Let $M(x)$ be the space of finite signed measures with $\|\mu\| = |\mu|(X)$ (total variation norm).

Theorem 3

Let X be a compact metric space. Then, for all $\ell \in [C(X)]^*$, there exists a unique $\mu \in M(X)$ such that $\ell(f) = \int f d\mu, \quad f \in C(X)$. Moreover, $\|\ell\| = \|\mu\|$.

Proof. Let $\ell \in [C(X)]^*$. By Lemma 2, $\ell = \ell_1 - \ell_2$ with ℓ_1, ℓ_2 positive. By Theorem 1, there exists finite regular measures μ_1, μ_2 such that for all $f \in C(X)$:

$$\ell_1(f) = \int f\, d\mu_1, \quad \ell_2(f) = \int f\, d\mu_2$$

$$\implies \ell(f) = \int f\, d(\mu_1 - \mu_2) = \int f\, d\mu, \quad f \in C(X)$$

Then, we could prove μ is a finite regular signed measure and is unique following a similar argument as Corollary 2 proof. Moreover, by Proposition 5, $\|\ell\| = \|\mu\|$.

Schwartz Space of Functions with Rapid Decrease

From Fourier Series to Fourier Transform

1. Let $T > 0$. Then, $\{\frac{1}{\sqrt{2\pi}} e^{\frac{in\pi}{T}x},\, n = 0, \pm 1, \dots\}$ is $2T$-periodic.
   This forms a CONS in $L^2([-T,T])$ (complex-valued functions).

2. If $h \in L^2([-T,T])$, then

$$h(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi}{T}x} \text{ in } L^2([-T,T]) \text{ with } c_n = \frac{1}{2T}\int_{-T}^{T} h(x)e^{-\frac{in\pi}{T}x}\,dx.$$

Also, $\displaystyle\int_{-T}^{T} |h(x)|^2\,dx = 2T\sum_{n=-\infty}^{\infty} |c_n|^2$.

Proof. First, let's compute the L^2 norm squared of h:

$$\int_{-T}^{T} |h(x)|^2\,dx = \int_{-T}^{T} h(x)\overline{h(x)}\,dx$$

Substitute the Fourier series for $h(x)$ and $\overline{h(x)}$:

$$\int_{-T}^{T}\left(\sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi}{T}x}\right)\left(\sum_{m=-\infty}^{\infty} \overline{c_m} e^{-\frac{im\pi}{T}x}\right)\,dx$$

Rearranging, we get:

$$\int_{-T}^{T}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} c_n\overline{c_m} e^{\frac{i(n-m)\pi}{T}x}\,dx$$

By Fubini's theorem,

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} c_n\overline{c_m}\int_{-T}^{T} e^{\frac{i(n-m)\pi}{T}x}\,dx$$

The exponential functions form an orthogonal system. When $n \neq m$,

$$\int_{-T}^{T} e^{\frac{i(n-m)\pi}{T}x}\,dx = \left[\frac{T}{i\pi(n-m)}e^{\frac{i(n-m)\pi}{T}x}\right]_{-T}^{T} = 0$$

When $n = m$,

$$\int_{-T}^{T} e^0\,dx = \int_{-T}^{T} 1\,dx = 2T$$

Therefore,

$$\sum_{n=-\infty}^{\infty} c_n\overline{c_n}\cdot 2T = 2T\sum_{n=-\infty}^{\infty} |c_n|^2$$

3. Let $f \in C_c^\infty(\mathbb{R})$, $f(x) = 0$ if $|x| > R > 0$. Then, for $T > R+1$, $|x| < T$,

$$f(x) = \sum_{n=-\infty}^{\infty}\frac{1}{2T}\int_{-T}^{T} f(y)e^{-\frac{in\pi}{T}y}\,dy\,e^{\frac{in\pi}{T}x} = \sum_{n=-\infty}^{\infty}\frac{1}{2T}\int_{-\infty}^{\infty} f(y)e^{-i\frac{n}{2T}2\pi y}\,dy\,e^{i\frac{n}{2T}2\pi x}.$$

Let $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(y)e^{-i2\pi\xi y}\,dy$, $\xi \in \mathbb{R}$. Note that $\hat{f} \in C_c^\infty(\mathbb{R})$. Hence,

$$f(x) = \sum_{n=-\infty}^{\infty}\frac{1}{2T}\hat{f}\left(\frac{n}{2T}\right)e^{i\frac{n}{2T}2\pi x}, \quad |x| < T, \text{ for all } T > R+1$$

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-T}^{T} |f(x)|^2\,dx = \frac{1}{2T}\sum_{n=-\infty}^{\infty} \left|\hat{f}\left(\frac{n}{2T}\right)\right|^2 \text{ for } T > R+1.$$

For any $x \in \mathbb{R}$,

$$f(x) = \lim_{T\to\infty} \sum_{n=-\infty}^{\infty} \frac{1}{2T}\hat{f}\left(\frac{n}{2T}\right) e^{i2\pi \frac{n}{2T} x} = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{i2\pi\xi x}\,d\xi.$$

Also,

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \frac{1}{2T}\sum_{n=-\infty}^{\infty} \left|\hat{f}\left(\frac{n}{2T}\right)\right|^2 = \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2\,d\xi \text{ for all } T > R+1.$$

For finite $T > R+1$, we established:

$$\int_{-T}^{T} |f(x)|^2\,dx = \frac{1}{2T}\sum_{n=-\infty}^{\infty} \left|\hat{f}\left(\frac{n}{2T}\right)\right|^2$$

Since f has compact support within $[-R, R]$, when $T > R+1$,

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-T}^{T} |f(x)|^2\,dx$$

$$\sum_{n=-\infty}^{\infty} \left|\hat{f}\left(\frac{n}{2T}\right)\right|^2 \cdot \frac{1}{2T} = \sum_{n=-\infty}^{\infty} |\hat{f}(\xi_n)|^2 \cdot \Delta\xi$$

This is precisely the form of a Riemann sum for the function $|\hat{f}(\xi)|^2$ over the entire real line:

$$\sum_{n=-\infty}^{\infty} |\hat{f}(\xi_n)|^2 \cdot \Delta\xi$$

As $T \to \infty$, the spacing $\Delta\xi = \frac{1}{2T} \to 0$, making the partition finer. Under appropriate conditions (which $\hat{f}$ satisfies since it comes from a compactly supported $f \in C_c^\infty(\mathbb{R})$), the Riemann sum converges to the integral:

$$\lim_{T\to\infty} \sum_{n=-\infty}^{\infty} |\hat{f}(\xi_n)|^2 \cdot \Delta\xi = \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2\,d\xi$$

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{i2\pi\xi x}\,d\xi, \quad x \in \mathbb{R} \text{ and } \int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2\,d\xi$$

Fourier transform of $f \in C_c^\infty(\mathbb{R})$:

$$\hat{f}(\xi) = (\mathcal{F}f)(\xi) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi\xi x}\,dx, \quad \xi \in \mathbb{R}$$

The inverse Fourier transform of $g \in C_c^\infty(\mathbb{R})$:

$$(\mathcal{F}^{-1}g)(x) = \int_{-\infty}^{\infty} g(\xi) e^{i2\pi\xi x}\, d\xi, \quad x \in \mathbb{R}.$$

We also have:

$$\mathcal{F}^{-1}\mathcal{F}f = f.$$

Note that $(\mathcal{F}^{-1}f)(x) = \hat{f}(-x)$, for any x. Also, $\mathcal{F}$ is "unitary". It turns out $F(C_c^\infty(\mathbb{R})) \subseteq S(\mathbb{R})$, the space of functions of rapid decrease. Moreover, $F : S(\mathbb{R}) \to S(\mathbb{R})$ is a bijection.

Space of functions of Rapid Decrease

$$S(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) : \sup_x (1+|x|)^N |f^{(n)}(x)| < \infty \text{ for any } N, n \in \mathbb{Z} \text{ and } N, n \geq 0\}.$$

Functions of Rapid Decrease

Let $f, g : \mathbb{R}^d \to \mathbb{C}$. For $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$, where $\mathbb{N}_0 = \{0, 1, 2 \cdots\}$,

$$|\alpha| = \alpha_1 + \cdots + \alpha_d,$$

$$\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}, \text{ total derivative of order } |\alpha|,$$

$$\alpha! = \alpha_1! \cdots \alpha_d!, \quad 0! = 1.$$

Product Rule

$$\partial^\alpha(fg) = \sum_{\beta \leq \alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} \partial^\beta f \partial^{\alpha-\beta} g,$$

where $\beta \leq \alpha \iff \beta_k \leq \alpha_k$ for any k.

$$x^\alpha := x_1^{\alpha_1} \cdots x_d^{\alpha_d}, \quad \text{where } \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d,$$

$$x = (x_1, \ldots, x_d) \in \mathbb{R}^d.$$

Definition of Schwartz Space

For $\alpha \in \mathbb{N}_0^d$ and $N \in \mathbb{N}_0$,

$$|f|_{N,\alpha} = \sup_{x \in \mathbb{R}^d} (1+|x|)^N |\partial^\alpha f(x)|.$$

The Schwartz space of functions of rapid decrease is defined as:

$$S = S(\mathbb{R}^d) = \{f \in C^\infty(\mathbb{R}^d) : |f|_{N,\alpha} < \infty \text{ for any } N, \alpha\}.$$

Properties

1. If $|f|_{N,\alpha} < \infty$ if and only if $(1+|x|)^N |f(x)| \leq C_{\alpha,N}$ for all $x \in \mathbb{R}^d$, then

$$|\partial^\alpha f(x)| \leq \frac{C_{\alpha,N}}{(1+|x|)^N} \text{ for any } x \in \mathbb{R}^d.$$

2. If $f \in S$, then

$$|\partial^\alpha f(x)| \leq \frac{C_{\alpha,N}}{(1+|x|)^N} \text{ for any } x \in \mathbb{R}^d.$$

3. $C_c^\infty(\mathbb{R}^d) \subseteq S(\mathbb{R}^d)$. Example: $e^{-|x|^2} \in S(\mathbb{R}^d)$ but $e^{-|x|} \notin C_c^\infty(\mathbb{R}^d)$ because it is non-zero everywhere.

Seminorms and Convergence

1. $|f|_{N,\alpha}$ with $f \in S$ is a seminorm.

$$|f+g|_{N,\alpha} \leq |f|_{N,\alpha} + |g|_{N,\alpha},$$
$$|af|_{N,\alpha} = |a||f|_{N,\alpha},$$
$$|f|_{N,\alpha} = 0 \text{ not implies } f = 0.$$

2. Let $f_k, f \in S$. We say $f_k \to f$ in S if

$$|f_k - f|_{N,\alpha} \to 0 \text{ for any } N, \alpha.$$

3. For $N \in \mathbb{N}_0$ and $f \in S$,

$$|f|_{(N)} = \max_{|\alpha| \leq N} |f|_{N,\alpha} = \max_{|\alpha| \leq N} \sup_x (1+|x|)^N |\partial^\alpha f(x)|$$

$|f|_{(N)}$ are norms on S, increasing in N, $|f|_{(N)} \leq |f|_{(N+1)}$.

4. For $f, g \in S$,

$$\rho(f,g) = \sum_{N=1}^\infty 2^{-N} \min(|f-g|_{(N)}, 1)$$

Properties of $S(\mathbb{R}^d)$

1. If $f \in S$ then $\partial^\alpha f \in S$.
2. If $f, g \in S$ then $fg \in S$.
3. If $f \in S$ then $Pf \in S$ for any polynomial P.
4. $f \in S \Rightarrow f \in L^p(\mathbb{R}^d)$ for any $p > 0$.

Proof. For $p = \infty$ this is easy. For $0 < p < \infty$,

$$\int_{\mathbb{R}^d} |f|^p \, dx = \int_{\mathbb{R}^d} (|f|(1+|x|)^{\frac{d+1}{p}})^p \frac{dx}{(1+|x|)^{d+1}} \leq |f|^p_{\lceil \frac{d+1}{p} \rceil, 0} \int_{\mathbb{R}^d} \frac{dx}{(1+|x|)^{d+1}} < \infty$$

5. If $f, g \in S$ then $f * g \in S$.

Proof. By definition of convolution,

$$\partial^\alpha (f * g)(x) = \int_{\mathbb{R}^d} \partial^\alpha f(x-y) g(y)\, dy$$

$$1 + |x| \le 1 + |x-y| + |y| \le (1 + |x-y|)(1+|y|)$$

$$\begin{aligned}
(1+|x|)^N |\partial^\alpha (f*g)(x)| &\le \int (1+|x-y|)^N |\partial^\alpha f(x-y)| (1+|y|)^N |g(y)|\, dy \\
&\le |f|_{N,\alpha} \int (1+|y|)^{N+d+1} |g(y)| \frac{dy}{(1+|y|)^{d+1}} \\
&\le |f|_{N,\alpha} |g|_{N+d+1,0} \int \frac{dy}{(1+|y|)^{d+1}} < \infty
\end{aligned}$$

Continuous Linear Functionals on S

Let $T : S(\mathbb{R}^d) \to \mathbb{C}$ be linear. Then, T is continuous with respect to the metric ρ on S if and only if

There exists $C > 0$ and $N \in \mathbb{N}_0$ such that $|T(f)| \le C|f|_{(N)}$ for all $f \in S$.

Proof. ($\Rightarrow$) Let T be continuous. Since T is continuous at zero, $|T(f)| = |T(f) - T(0)| \le 1$ if $\rho(f, 0) \le 2\delta$ for some $\delta > 0$.

$$\begin{aligned}
\rho(f,0) &= \sum_{N=1}^{\infty} 2^{-N} \min\{1, |f|_{(N)}\} \\
&\le \sum_{N=1}^{N_0} 2^{-N} |f|_{(N_0)} + \sum_{N=N_0+1}^{\infty} 2^{-N} \\
&\le |f|_{(N_0)} + 2^{-N_0}
\end{aligned}$$

Choose N_0 so that $2^{-N_0} \le \delta$. Let $g \in S$ and $g \ne 0$. Then, $f = \frac{\delta}{|g|_{(N_0)}} g$ has $|f|_{(N_0)} = \delta$.

$$\implies |f|_{(N_0)} \le \delta \Rightarrow \rho(f, 0) \le 2\delta \Rightarrow |T(f)| \le 1$$

$$\implies \sum_{N=1}^{\infty} 2^{-N} \min\{1, |f|_{(N)}\} \le |f|_{(N_0)} + 2^{-N_0} \le 2\delta \text{ for all } N_0 \in \mathbb{N}$$

$$|T(f)| = |T(g)| \cdot \frac{\delta}{|g|_{(N_0)}} \le 1 \Rightarrow |T(g)| \le \frac{1}{\delta} |g|_{(N_0)}$$

($\Leftarrow$) Let $|T(f)| \le C|f|_{(N_0)}, f \in S$. Let $f_n \to f$ in S. Then, when $|f_n - f|_{(N_0)} \to 0$,

$$|T(f_n) - T(f)| = |T(f_n - f)| \le C|f_n - f|_{(N_0)} \to 0$$

Corollary 0

A linear functional $T : S \to \mathbb{C}$ is continuous if and only if there exist $C > 0$, $\alpha_1, \ldots, \alpha_m \in \mathbb{N}_0^d$, and $N_1, \ldots, N_m \in \mathbb{N}_0$ such that:

$$|T(f)| \leq C \sum_{k=1}^{m} |f|_{N_k,\alpha_k}$$

Fourier Transform on $S(\mathbb{R}^d)$

(i) Given $f \in L^1(\mathbb{R}^d)$, its Fourier transform is:

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-i2\pi x\cdot\xi}\, dx, \quad \xi \in \mathbb{R}^d$$

where $x \cdot \xi = x_1\xi_1 + \cdots + x_d\xi_d$.

(ii) Given $g \in L^1(\mathbb{R}^d)$, its inverse Fourier transform is:

$$\mathcal{F}^{-1}g(x) = \int_{\mathbb{R}^d} g(\xi) e^{i2\pi x\cdot\xi}\, d\xi, \quad x \in \mathbb{R}^d$$

Note.

(a) $\mathcal{F}f(\xi)$ and $\mathcal{F}^{-1}g(x)$ are bounded, continuous functions.

(b) $\mathcal{F}$ and $\mathcal{F}^{-1}$ are linear operators.

(c) $|\mathcal{F}f(\xi)| \leq \int_{\mathbb{R}^d} |f|\, dx = |f|_{L^1}$.

(d) $|\mathcal{F}^{-1}g(x)| \leq |g|_{L^1}$.

(e) $\mathcal{F}^{-1}g(x) = \mathcal{F}g(-x)$, $x \in \mathbb{R}^d$.

(f) $\widehat{\partial^\alpha f} = (i2\pi\xi)^\alpha \hat{f}$.

Simple Properties of Fourier Transform

Property 1

If $f \in S$, then for $\alpha \in \mathbb{N}_0^d$,

$$\widehat{\partial^\alpha f}(\xi) = (i2\pi\xi)^\alpha \hat{f}(\xi) = (i2\pi)^{|\alpha|} \xi_1^{\alpha_1} \cdots \xi_d^{\alpha_d} \hat{f}(\xi)$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{N}_0^d$, $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d}$, and $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$.

Proof. For $d = 1$ and $\alpha = 1$,

$$\widehat{f^{(1)}}(\xi) = i2\pi\xi \hat{f}(\xi), \quad \xi \in \mathbb{R}.$$

By definition,

$$\begin{aligned}\widehat{f^{(1)}}(\xi) &= \int_{-\infty}^{\infty} f'(x)e^{-i2\pi x\xi}\,dx = \lim_{n\to\infty}\int_{-n}^{n} f'(x)e^{-i2\pi x\xi}\,dx \\ &= \lim_{n\to\infty}\left[f(x)e^{-i2\pi x\xi}\right]_{-n}^{n} - (-i2\pi\xi)\int_{-n}^{n} f(x)e^{-i2\pi x\xi}\,dx \\ &= 0 + i2\pi\xi\hat{f}(\xi) = i2\pi\xi\hat{f}(\xi)\end{aligned}$$

Example 1

Let $\Delta u(x) = \frac{\partial^2}{\partial x_1^2}u(x) + \cdots + \frac{\partial^2}{\partial x_d^2}u(x)$, where $u \in S$. Find $\widehat{\Delta u}(\xi)$.

$\frac{\partial^2}{\partial x_1^2} = \partial^\alpha$ with $\alpha = [2, 0, 0, \cdots 0]$. By property 1,

$$\begin{aligned}\widehat{\Delta u}(\xi) &= \widehat{\frac{\partial^2}{\partial x_1^2}u}(\xi) + \cdots + \widehat{\frac{\partial^2}{\partial x_d^2}u}(\xi) \\ &= (i2\pi\xi_1)^2\hat{u}(\xi) + \cdots + (i2\pi\xi_d)^2\hat{u}(\xi) \\ &= \hat{u}(\xi)\left(-4\pi^2(\xi_1^2 + \xi_2^2 + \cdots + \xi_d^2)\right) \\ &= -4\pi^2|\xi|^2\hat{u}(\xi)\end{aligned}$$

Property 2

1. Let $f \in S$, $y \in \mathbb{R}^d$. Then:

$$\widehat{f(\cdot + y)}(\xi) = e^{i2\pi\xi\cdot y}\hat{f}(\xi)$$

Proof.

$$\begin{aligned}\widehat{f(\cdot + y)}(\xi) &= \int_{\mathbb{R}^d} f(x+y)e^{-i2\pi x\cdot\xi}\,dx \\ &= \int_{\mathbb{R}^d} f(z)e^{-i2\pi(z-y)\cdot\xi}\,dz \quad (\text{where } z = x+y \text{ and } dz = dx) \\ &= e^{i2\pi y\cdot\xi}\hat{f}(\xi)\end{aligned}$$

2. Let $f \in S$, $\delta \in \mathbb{R}$ and $\epsilon \neq 0$. Then,

$$\widehat{f(\delta\cdot)}(\xi) = |\delta|^{-d}\hat{f}\left(\frac{\xi}{\delta}\right)$$

Proof.

$$\begin{aligned}\widehat{f(\delta\cdot)}(\xi) &= \int_{\mathbb{R}^d} f(\delta x)e^{-i2\pi x\cdot\xi}\,dx \\ &= \int_{\mathbb{R}^d} f(z)e^{-i2\pi\frac{z\cdot\xi}{\delta}}\delta^{-d}\,dz \quad (\text{where } z = \delta x \text{ and } dz = \delta^d dx) \\ &= \delta^{-d}\hat{f}\left(\frac{\xi}{\delta}\right)\end{aligned}$$

Example 2

Let $f \in S$, $\int f\,dx \neq 0$. Show that:

$$\lim_{\delta\to 0} |\widehat{f(\delta\cdot)}(\xi)| = \begin{cases} 0, & \xi \neq 0 \\ +\infty, & \xi = 0 \end{cases}$$

Assume $\delta \neq 0$.

$$|\widehat{f(\delta\cdot)}(\xi)| \overset{\delta\neq 0}{=} |\delta|^{-d} \left| \hat{f}\left(\frac{\xi}{\delta}\right) \right|$$

Since $\hat{f} \in S$, it satisfies:

$$|\hat{f}(\xi)| \leq \frac{C}{(1+|\xi|)^{d+1}},$$

where $C = |\hat{f}|_{d+1,0}$ is a seminorm. When $\xi \neq 0$,

$$\begin{aligned} |\widehat{f(\delta\cdot)}(\xi)| &= \frac{1}{|\xi|^d} \cdot \left(\frac{|\xi|}{|\delta|}\right)^d \cdot |\hat{f}(\xi)| \\ &\leq \frac{1}{|\xi|^d} \cdot \left(\frac{|\xi|}{|\delta|}\right)^d \cdot \frac{|\hat{f}|_{d+1,0}}{(1+|\xi|/|\delta|)^{d+1}} \to 0 \quad \text{as} \quad \delta \to 0 \end{aligned}$$

When $\xi = 0$,

$$|\widehat{f(\delta\cdot)}(0)| = |\delta|^{-d} |\hat{f}(0)| \to \infty \quad \text{as} \quad \delta \to 0$$

Note that $\hat{f}(0) = \int f(x)\,dx \neq 0$, so $|\hat{f}(0)| \neq 0$. On the other hand, $\lim_{\delta\to 0} f(\delta x) = f(0)$ for $f \in S$. Note the opposite tendencies of $f(\delta x)$ and $\widehat{f(\delta\cdot)}(\xi)$ as $\delta \to 0$.

Property 3

If $f \in S$, then $\hat{f} \in C^\infty$ with all derivatives bounded, and:

$$\partial^\beta \hat{f}(\xi) = \hat{g}(\xi) \quad \text{with} \quad g(x) = (-i2\pi x)^\beta f(x) \in S, \quad \beta \in \mathbb{N}_0^d.$$

Proof. (See Folland 2-27)

$$\partial^\beta \hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) \frac{\partial^\beta}{\partial \xi} e^{-i2\pi x\cdot\xi}\,dx = \int_{\mathbb{R}^d} \underbrace{f(x)(-i2\pi x)^\beta}_{g(x)} e^{-i2\pi x\cdot\xi}\,dx = \hat{g}(\xi)$$

Proposition 1

The Fourier transform F maps S to S. We write $F : S \to S$.

Proof. Let $f \in S$. By Property 3 above, $\hat{f} \in C^\infty$. Then, $\hat{f} \in S$ if and only if $\sup_{\xi\in\mathbb{R}^d} |\xi^\alpha \partial^\beta \hat{f}(\xi)| < \infty$ for any $\alpha, \beta \in \mathbb{N}_0^d$.

$$\begin{aligned} \xi^\alpha \partial^\beta \hat{f}(\xi) &= (i2\pi)^{-|\alpha|} (i2\pi\xi)^\alpha \partial^\beta \hat{f}(\xi) \\ &= (i2\pi)^{-|\alpha|} (i2\pi\xi)^\alpha \hat{g}(\xi) \text{ with } g(x) = (-i2\pi x)^\beta f(x) \in S \end{aligned}$$

Hence,

$$\sup_{\xi\in\mathbb{R}^d} |\xi^\alpha \partial^\beta \hat{f}(\xi)| = (2\pi)^{-|\alpha|} \sup_{\xi\in\mathbb{R}^d} \widehat{\partial^\alpha g}(\xi) \leq (2\pi)^{-|\alpha|} |\partial^\alpha g|_{L_1} < \infty$$

Proposition 2

Let $f, g \in S$. Then, $\widehat{f * g}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$.

Proof. By Fubini's Theorem,

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x-y)g(y)\,dy\,e^{-i2\pi x\cdot\xi}\,dx \\
&= \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} f(x-y)\,e^{-i2\pi(x-y)\cdot\xi}\,dx\right)\cdot g(y)\,e^{-i2\pi y\cdot\xi}\,dy\,, \quad \text{set } z = x-y \text{ and } dz = dx \\
&= \int_{\mathbb{R}^d}\left(\hat{f}(\xi)\right)\cdot g(y)\,e^{-i2\pi y\cdot\xi}\,dy = \hat{f}(\xi)\cdot\hat{g}(\xi)
\end{aligned}$$

We can swap integrals because $\iint |f(x-y)|\cdot|g(y)|\,dy\,dx = |f|_{L^1}\cdot|g|_{L^1} < \infty$.

Proposition 3

Let $f, g \in S$. Then,

$$\int_{\mathbb{R}^d}\hat{f}(z)g(z)\,dz = \int_{\mathbb{R}^d} f(y)\hat{g}(y)\,dy$$

$$\int_{\mathbb{R}^d}\mathcal{F}^{-1}(f)(z)g(z)\,dz = \int_{\mathbb{R}^d} f(y)\mathcal{F}^{-1}(g)(y)\,dy$$

.

Proof. The proof will use Fubini's Theorem. Let $f, g \in S$. Then,

$$\begin{aligned}
\int_{\mathbb{R}^d}\hat{f}(z)g(z)\,dz &= \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} f(y)\,e^{-i2\pi y\cdot z}\,dy\right)g(z)\,dz \\
&= \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} g(z)\,e^{-i2\pi y\cdot z}\,dz\right)f(y)\,dy \\
&= \int_{\mathbb{R}^d}\hat{g}(y)f(y)\,dy
\end{aligned}$$

We can use Fubini's Theorem because:

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|f(y)|\cdot|g(y)|\cdot|e^{i2\pi y\cdot z}|\,dy\,dz = |f|_{L^1}\cdot|g|_{L^1} < \infty$$

since $f, g \in L^1$ since $f, g \in S$. Analogous argument proves the $\mathcal{F}^{-1}$ case.

Inverse of $\mathcal{F}$

Theorem 1

Let $f \in S$. Then, $\mathcal{F}^{-1}\mathcal{F}(f)(x) = \mathcal{F}\mathcal{F}^{-1}(f)(x)$ for $x \in \mathbb{R}^d$. In particular,

$$\mathcal{F}^{-1}(\hat{f})(x) = \int_{\mathbb{R}^d}\hat{f}(\xi)e^{i2\pi x\cdot\xi}\,d\xi = f(x) \text{ for } x \in \mathbb{R}^d.$$

This means both $\mathcal{F}, \mathcal{F}^{-1} : S \to S$ are bijections.

Proof. Let $f \in S$. We first prove that inversion holds at $x = 0$.

$$f(0) = \int_{\mathbb{R}^d} \hat{f}(\xi)\, d\xi.$$

For $\delta > 0$ and $k(x) = e^{-\pi|x|^2}$,

$$\int_{\mathbb{R}^d} f(\delta y)\hat{k}(y)\, dy \longrightarrow f(0) \text{ as } \delta \to 0 \text{ by dominant convergence theorem.}$$

By Proposition 3,

$$\begin{aligned}
\int_{\mathbb{R}^d} f(\delta y)\hat{k}(y)\, dy &= \int \widehat{f(\delta \cdot)}(\xi) k(\xi)\, d\xi \\
&= \int \delta^{-d} \hat{f}\left(\frac{\xi}{\delta}\right) k(\xi)\, d\xi \text{ (by Property 2(b))} \\
&= \int \delta^{-d} \hat{f}(z) k(\delta z) \delta^d\, dz \text{ where } z = \frac{\xi}{\delta} \text{ and } d\xi = \delta^d dz \\
&= \int \hat{f}(z) k(\delta z)\, dz
\end{aligned}$$

By dominant convergence theorem, when $\delta \to 0$, $\int \hat{f}(z)k(\delta z)\, dz \to \int \hat{f}(z)k(0)\, dz = \int \hat{f}(z)\, dz$ as $k(0) = 1$ and $\int \hat{f}(z)\, dz = f(0)$. In general case, let $y \in \mathbb{R}^d$ and consider $f(x + y)$ for $x \in \mathbb{R}^d$. Note that $f(y) = f(0 + y)$. Using the result that inversion holds at 0,

$$\begin{aligned}
f(y) = f(0 + y) &= \int \mathcal{F}(f(\cdot + y))(\xi)\, d\xi \\
&= \int \hat{f}(\xi) e^{i2\pi \xi \cdot y}\, d\xi \text{ (by property 2(a))} \\
&= \mathcal{F}^{-1}(\hat{f})(y)
\end{aligned}$$

We have shown $\mathcal{F}^{-1}\mathcal{F}(f) = f$ so we now show $\mathcal{F}\mathcal{F}^{-1}(f) = f$. Recall that $\mathcal{F}(g)(x) = \mathcal{F}^{-1}(g)(-x)$. Now $\mathcal{F}\mathcal{F}^{-1}(f)(x) = \mathcal{F}^{-1}(\mathcal{F}^{-1}(f))(-x)$ and:

$$\mathcal{F}^{-1} f(x) = \int f(\xi) e^{i2\pi x \cdot \xi}\, d\xi = \int f(-z) e^{-i2\pi z \cdot x}\, dz = \mathcal{F}\tilde{f}(x)$$

By the change of variable, $\xi = -z$ and $\tilde{f}(x) = f(-x)$ for $x \in \mathbb{R}^d$. Therefore,

$$\mathcal{F}(\mathcal{F}^{-1}(f))(x) = \mathcal{F}^{-1}(\mathcal{F}^{-1}(f))(-x) = \mathcal{F}^{-1}(\mathcal{F}(\tilde{f}))(-x) = \tilde{f}(-x) = f(x).$$

Thus, we prove that $\mathcal{F}\mathcal{F}^{-1}(f)(x) = f(x) = \mathcal{F}^{-1}\mathcal{F}(f)(x)$.

Remark.

1. First Use of DCT in the Proof.

$$\int_{\mathbb{R}^d} f(\delta y)\hat{k}(y)\, dy \longrightarrow f(0) \quad \text{as } \delta \to 0.$$

- *Pointwise Convergence.* Since f is continuous (because $f \in S$, the Schwartz space), $f(\delta y) \to f(0)$ as $\delta \to 0$ for each y.
- *Dominating Function.* Since $\hat{f} \in S$, it is rapidly decreasing, meaning $|\hat{f}(\xi)| \leq \frac{C}{(1+|\xi|)^{d+1}}$ for some $C > 0$. Meanwhile, $|k(\delta\xi)| \leq 1$ (since k is a Gaussian). Thus, $|\hat{f}(\xi)k(\delta\xi)| \leq |\hat{f}(\xi)|$, and $\hat{f}(\xi)$ is integrable (because $\hat{f} \in S$).

DCT allows us to move the limit inside:

$$\lim_{\delta\to 0}\int f(\delta y)\hat{k}(y)\,dy = \int \lim_{\delta\to 0} f(\delta y)\hat{k}(y)\,dy = \int f(0)\hat{k}(y)\,dy = f(0).$$

2. Second Use of DCT in the Proof.

$$\int \hat{f}(\xi)k(\delta\xi)\,d\xi \longrightarrow \int \hat{f}(\xi)\,d\xi \quad \text{as } \delta \to 0.$$

- *Pointwise Convergence.* $k(\delta\xi) = e^{-\pi\delta^2|\xi|^2} \to 1$ as $\delta \to 0$ for each ξ.
- *Dominating Function.* Since $\hat{f} \in S$, it is rapidly decreasing, meaning $|\hat{f}(\xi)| \leq \frac{C}{(1+|\xi|)^{d+1}}$ for some $C > 0$. Meanwhile, $|k(\delta\xi)| \leq 1$ since k is a Gaussian. Thus, $|\hat{f}(\xi)k(\delta\xi)| \leq |\hat{f}(\xi)|$, and $\hat{f}(\xi)$ is integrable because $\hat{f} \in S$.

DCT allows us to move the limit inside:

$$\lim_{\delta\to 0}\int \hat{f}(\xi)k(\delta\xi)\,d\xi = \int \hat{f}(\xi)\lim_{\delta\to 0} k(\delta\xi)\,d\xi = \int \hat{f}(\xi)\,d\xi.$$

3. Why the Schwartz Space S is Important?
   - Functions in S decay rapidly: Both f and $\hat{f}$ decrease faster than any polynomial, ensuring integrability.
   - Gaussians are nice dominating functions: $k(x) = e^{-\pi|x|^2}$ is smooth, integrable, and dominates the integrals involved.

If f were not in S, we might not have a suitable dominating function.

Plancherel's Equality

For $f \in S$, $\int |f(x)|^2\,dx = \int |\hat{f}(\xi)|^2\,d\xi$, so $|f|_{L^2} = |\hat{f}|_{L^2}$ where $L^2 = L^2(\mathbb{R}^d)$. *Idea of the proof:* Use $f(0) = \int \hat{f}(\xi)\,d\xi$ and the Fourier transform of the convolution: Let $(f * g)(x) = \int f(x-y)g(y)\,dy$ and $\mathcal{F}(f*g)(x) = \hat{f}(\xi)\hat{g}(\xi)$. Applying Proposition 2 and $f(0) = \int \hat{f}(\xi)\,d\xi$,

$$(f*g)(0) = \int g(y)f(-y)\,dy = \int \mathcal{F}(f*g)(\xi)\,d\xi = \int \hat{f}(\xi)\hat{g}(\xi)\,d\xi \quad (\#)$$

In (#) above, we want to take $f = g$ and deal with the negative y part.

Proposition 4 (Plancherel's Equality)

For any $f, g \in S$, $\int g(y)\overline{f(y)}\,dy = \int \hat{g}(\xi)\overline{\hat{f}(\xi)}\,d\xi$. Thus, $\langle g, f\rangle_{L^2} = \langle \hat{g}, \hat{f}\rangle_{L^2}$. In particular, $|f|_{L^2} = |\hat{f}|_{L^2}$ in the case $f = g$.

Proof. Let $f, g \in S$. Set $\tilde{f}(y) = \overline{f(-y)}$ for $y \in \mathbb{R}^d$. Note that $\tilde{f} \in S$. By (#),

$$\int g(y)\overline{f(y)}\,dy = \int g(y)\tilde{f}(-y)\,dy = \int \hat{g}(\xi)\mathcal{F}(\tilde{f})(\xi)\,d\xi$$

Now, we compute $\mathcal{F}(\tilde{f})(\xi)$.

$$\begin{aligned}\mathcal{F}(\tilde{f})(\xi) &= \int \overline{f(-y)}e^{-i2\pi y\cdot\xi}\,dy = \int \overline{f(x)}e^{i2\pi x\cdot\xi}\,dx \quad \text{where } x = -y,\ dx = dy\,|J| = dy \\ &= \overline{\int f(x)e^{-i2\pi x\cdot\xi}\,dx} = \overline{\hat{f}(\xi)}\end{aligned}$$

Hence, $\int g(y)\overline{f(y)}\,dy = \int \hat{g}(\xi)\overline{\hat{f}(\xi)}\,d\xi \iff \langle g, f\rangle_{L^2} = \langle \hat{g}, \hat{f}\rangle_{L^2}$.

Remark. $\mathcal{F}$ and $\mathcal{F}^{-1}$ can be extended to unitary maps on L^2. For $f \in L^2$, there exists $f_n \in S$ with $f_n \to f$ in L^2 since $C_c^\infty \subseteq S$ and C_c^∞ is dense in L^2. Then, by Proposition 4, if $|f_n - f_m|_{L^2} \to 0$ as $m, n \to \infty$, then

$$|f_n - f_m|_{L^2} = |\hat{f}_n - \hat{f}_m|_{L^2} = |\mathcal{F}^{-1}(f_n) - \mathcal{F}^{-1}(f_m)|_{L^2}$$

We define $\mathcal{F}f = \lim_{n\to\infty}\mathcal{F}(f_n)$ and $\mathcal{F}^{-1}f = \lim_{n\to\infty}\mathcal{F}^{-1}(f_n)$. This definition does not depend on the choice of the sequence f_n. Indeed, let $g_n \in S$ such that $g_n \to f$ in L^2. Then, $|g_n - f_n|_{L^2} \to 0$ as $n \to \infty$. Therefore, by proposition 4,

$$|g_n - f_n|_{L^2} = |\hat{g}_n - \hat{f}_n|_{L^2} = |\mathcal{F}^{-1}g_n - \mathcal{F}^{-1}f_n|_{L^2}$$

$$\implies \hat{g}_n \to \mathcal{F}f \text{ and } \mathcal{F}^{-1}g_n \to \mathcal{F}^{-1}f_n \text{ in } L^2$$

Hence, $\mathcal{F} : L^2 \to L^2$ is unitary with $\mathcal{F}^* = \mathcal{F}^{-1}$.

Corollary 1

$\mathcal{F}, \mathcal{F}^{-1} : S \to S$ can be extended to $\mathcal{F}, \mathcal{F}^{-1} : L^2 \to L^2$ as

$$\mathcal{F}f = \hat{f} = \lim_{n\to\infty}\mathcal{F}f_n,\ \mathcal{F}^{-1}f = \lim_{n\to\infty}\mathcal{F}^{-1}f_n \text{ in } L^2 \text{ with } f_n \in S,\ f_n \to f \text{ in } L^2.$$

We have $\mathcal{F}\mathcal{F}^{-1}f = \mathcal{F}^{-1}\mathcal{F}f = f$ for any $f \in L^2$. Also,

$$|\mathcal{F}f|_{L^2} = |\mathcal{F}^{-1}f|_{L^2} = |f|_{L^2}, \quad f \in L^2.$$

Proposition 6

Let $f \in L^1 \cap L^2$. Let $\hat{f} = \lim_{n\to\infty} \hat{f}_n, \mathcal{F}^{-1} f = \lim_{n\to\infty} \mathcal{F}^{-1} f_n$ in L^2 with $f_n \in S$ and $f_n \to f$ in L^2. Then, $\hat{f}(\xi) = \int f(x) e^{-i2\pi x\cdot\xi}\,dx$ with $\xi \in \mathbb{R}^d$. $\mathcal{F}^{-1} f(x) = \int f(\xi) e^{i2\pi x\cdot\xi}\,d\xi$ with $x \in \mathbb{R}^d$.

Proof. By Proposition 3, for any $h \in S$,

$$\int \hat{f}_n h = \int f_n \hat{h} \quad \text{and} \quad \int \mathcal{F}^{-1}(f_n) h = \int f_n \mathcal{F}^{-1}(h)$$

As $n \to \infty$,

$$\int \hat{f}_n h \to \int \hat{f} h, \int f_n \hat{h} \to \int f \hat{h}, \int \mathcal{F}^{-1}(f_n) h \to \int \mathcal{F}^{-1}(f) h, \int f_n \mathcal{F}^{-1}(h) \to \int f \mathcal{F}^{-1}(h)$$

$$\implies \int \hat{f} h = \int f \hat{h}, \quad \int \mathcal{F}^{-1}(f) h = \int f \mathcal{F}^{-1}(h)$$

Now, for all $h \in S$,

$$\begin{aligned}
\int \hat{f}(y) h(y)\,dy &= \int f(x) \hat{h}(x)\,dx \\
&= \int f(x) \left(\int h(y) e^{-i2\pi x\cdot y}\,dy \right) dx \\
&= \int \left(f(x) e^{-i2\pi x\cdot y}\,dx \right) h(y)\,dy \quad \text{(By Fubini's Theorem)}
\end{aligned}$$

Thus,

$$\hat{f}(y) = \int f(x) e^{-i2\pi x\cdot y}\,dx \quad \text{a.e.}$$

Fubini holds since $\iint |f(x)| \cdot |h(y)| \cdot |e^{-i2\pi x\cdot y}|\,dx\,dy = |f|_{L^1} \cdot |h|_{L^1} < \infty$. An analogous proof holds for $\mathcal{F}^{-1}$.

Corollary 2

Let $f \in L^2$. Then,

$$\mathcal{F} f(\xi) = \hat{f}(\xi) = \lim_{n\to\infty} \int_{|x|\le n} f(x) e^{-i2\pi x\cdot\xi}\,dx \text{ in } L^2,$$

$$\mathcal{F}^{-1} f(x) = \lim_{n\to\infty} \int_{|\xi|\le n} f(\xi) e^{i2\pi x\cdot\xi}\,d\xi \text{ in } L^2.$$

Proof. Since $f \in L^2$, we can approximate it by truncating it to the ball $|x| \le n$. Define:

$$f_n(x) = f(x) \cdot \chi_{|x|\le n},$$

where $\chi_{|x|\le n}$ is the indicator function of the ball of radius n. Then, f_n has compact support since it vanishes outside $|x| \le n$. By Hölder's inequality, and $f_n \in L^2$ since $f \in L^2$,

$$\int |f_n| \le \left(\int |f_n|^2 \right)^{1/2} \cdot \left(\int \chi_{|x|\le n} \right)^{1/2} < \infty$$

Therefore, $f_n \in L^1 \cap L^2$.

$f_n \to f$ in L^2 as $n \to \infty$ by the dominated convergence theorem, since $|f_n| \le |f|$ and $|f|^2$ is integrable. For $f_n \in L^1 \cap L^2$,

$$\mathcal{F}f(\xi) = \hat{f}_n(\xi) = \int_{\mathbb{R}^d} f_n(x) e^{-i2\pi x\cdot\xi}\, dx = \int_{|x|\le n} f(x) e^{-i2\pi x\cdot\xi}\, dx.$$

This is well-defined pointwise since $f_n \in L^1$. Since $f_n \to f$ in L^2, and the Fourier transform $\mathcal{F}$ is a unitary operator on L^2, we have:

$$\hat{f}_n \to \hat{f} \quad \text{in } L^2 \text{ as } n \to \infty.$$

This is because $|\hat{f}_n - \hat{f}|_{L^2} = |f_n - f|_{L^2} \to 0$ by Plancherel's Equality. Now, define:

$$g_n(\xi) = f(\xi) \cdot \chi_{|\xi|\le n}.$$

Then $g_n \to f$ in L^2. By the continuity of $\mathcal{F}^{-1}$ on L^2, we have $\mathcal{F}^{-1}g_n \to \mathcal{F}^{-1}f \quad \text{in } L^2$, which gives the second formula:

$$\mathcal{F}^{-1}f(x) = \lim_{n\to\infty} \int_{|\xi|\le n} f(\xi) e^{i2\pi x\cdot\xi}\, d\xi \quad (\text{in } L^2).$$

Recall. $f \in S$. Then, $\mathcal{F}^{-1}\mathcal{F}(f)(x) = \mathcal{F}\mathcal{F}^{-1}(f)(x) = f(x)$, $x \in \mathbb{R}^d$. $\mathcal{F}, \mathcal{F}^{-1} : S \to S$ are bijective.

Plancherel's Equality

For $f \in S$, $\int |f(x)|^2\, dx = \int |\hat{f}(\xi)|^2\, d\xi$, so $|f|_{L^2} = |\hat{f}|_{L^2}$ where $L^2 = L^2(\mathbb{R}^d)$.

Idea of the proof: Use $f(0) = \int \hat{f}(\xi)\, d\xi$ and the Fourier transform of the convolution: Let $(f * g)(x) = \int f(x-y)g(y)\, dy$ and $\mathcal{F}(f*g)(x) = \hat{f}(\xi)\hat{g}(\xi)$. Applying Proposition 2 and $f(0) = \int \hat{f}(\xi)\, d\xi$,

$$(f*g)(0) = \int g(y)f(-y)\, dy = \int \mathcal{F}(f*g)(\xi)\, d\xi = \int \hat{f}(\xi)\hat{g}(\xi)\, d\xi \quad (\#).$$

In (#) above, we want to take $f = g$ and deal with the negative y part.

Proposition 4 (Plancherel's Equality)

For any $f, g \in S$, $\int g(y)\overline{f(y)}\, dy = \int \hat{g}(\xi)\overline{\hat{f}(\xi)}\, d\xi$. Thus, $\langle g, f\rangle_{L^2} = \langle \hat{g}, \hat{f}\rangle_{L^2}$. In particular, $|f|_{L^2} = |\hat{f}|_{L^2}$ in the case $f = g$.

Proof. Let $f, g \in S$. Set $\tilde{f}(y) = \overline{f(-y)}$ for $y \in \mathbb{R}^d$. Note that $\tilde{f} \in S$. By (#),

$$\int g(y)\overline{f(y)}\, dy = \int g(y)\tilde{f}(-y)\, dy = \int \hat{g}(\xi)\mathcal{F}(\tilde{f})(\xi)\, d\xi$$

Now, we compute $\mathcal{F}(\tilde{f})(\xi)$.

$$\mathcal{F}(\tilde{f})(\xi) = \int \overline{f(-y)} e^{-i2\pi y\cdot\xi}\, dy = \int \overline{f(x)} e^{i2\pi x\cdot\xi}\, dx \quad \text{where } x = -y,\ dx = dy\,|J| = dy$$
$$= \overline{\int f(x) e^{-i2\pi x\cdot\xi}\, dx} = \overline{\hat{f}(\xi)}$$

Hence, $\int g(y)\overline{f(y)}\, dy = \int \hat{g}(\xi)\overline{\hat{f}(\xi)}\, d\xi \iff \langle g, f\rangle_{L^2} = \langle \hat{g}, \hat{f}\rangle_{L^2}$.

Remark. $\mathcal{F}$ and $\mathcal{F}^{-1}$ can be extended to unitary maps on L^2. For $f \in L^2$, there exists $f_n \in S$ with $f_n \to f$ in L^2 since $C_c^\infty \subseteq S$ and C_c^∞ is dense in L^2. Then, by Proposition 4, if $|f_n - f_m|_{L^2} \to 0$ as $m, n \to \infty$, then

$$|f_n - f_m|_{L^2} = |\hat{f}_n - \hat{f}_m|_{L^2} = |\mathcal{F}^{-1}(f_n) - \mathcal{F}^{-1}(f_m)|_{L^2}$$

We define $\mathcal{F}f = \lim_{n\to\infty} \mathcal{F}(f_n)$ and $\mathcal{F}^{-1}f = \lim_{n\to\infty} \mathcal{F}^{-1}(f_n)$. This definition does not depend on the choice of the sequence f_n. Indeed, let $g_n \in S$ such that $g_n \to f$ in L^2. Then, $|g_n - f_n|_{L^2} \to 0$ as $n \to \infty$. Therefore, by proposition 4,

$$|g_n - f_n|_{L^2} = |\hat{g}_n - \hat{f}_n|_{L^2} = |\mathcal{F}^{-1}g_n - \mathcal{F}^{-1}f_n|_{L^2},$$
$$\implies \hat{g}_n \to \mathcal{F}f \text{ and } \mathcal{F}^{-1}g_n \to \mathcal{F}^{-1}f_n \text{ in } L^2.$$

Hence, $\mathcal{F} : L^2 \to L^2$ is unitary with $\mathcal{F}^* = \mathcal{F}^{-1}$.

Corollary 1

$\mathcal{F}, \mathcal{F}^{-1} : S \to S$ can be extended to $\mathcal{F}, \mathcal{F}^{-1} : L^2 \to L^2$ as

$$\mathcal{F}f = \hat{f} = \lim_{n\to\infty} \mathcal{F}f_n,\ \mathcal{F}^{-1}f = \lim_{n\to\infty} \mathcal{F}^{-1}f_n \text{ in } L^2 \text{ with } f_n \in S,\ f_n \to f \text{ in } L^2.$$

We have $\mathcal{F}\mathcal{F}^{-1}f = \mathcal{F}^{-1}\mathcal{F}f = f$ for any $f \in L^2$. Also,

$$|\mathcal{F}f|_{L^2} = |\mathcal{F}^{-1}f|_{L^2} = |f|_{L^2}, \quad f \in L^2.$$

Proposition 6

Let $f \in L^1 \cap L^2$. Let $\hat{f} = \lim_{n\to\infty} \hat{f}_n, \mathcal{F}^{-1}f = \lim_{n\to\infty} \mathcal{F}^{-1}f_n$ in L^2 with $f_n \in S$ and $f_n \to f$ in L^2. Then, $\hat{f}(\xi) = \int f(x)e^{-i2\pi x\cdot\xi}\, dx$ with $\xi \in \mathbb{R}^d$. $\mathcal{F}^{-1}f(x) = \int f(\xi)e^{i2\pi x\cdot\xi}\, d\xi$ with $x \in \mathbb{R}^d$.

Proof. By Proposition 3, for any $h \in S$,

$$\int \hat{f}_n h = \int f_n \hat{h} \quad \text{and} \quad \int \mathcal{F}^{-1}(f_n)h = \int f_n \mathcal{F}^{-1}(h)$$

As $n \to \infty$,

$$\int \hat{f}_n h \to \int \hat{f} h, \int f_n \hat{h} \to \int f \hat{h}, \int \mathcal{F}^{-1}(f_n)h \to \int \mathcal{F}^{-1}(f)h, \int f_n \mathcal{F}^{-1}(h) \to \int f \mathcal{F}^{-1}(h),$$

$$\implies \int \hat{f}h = \int f\hat{h}, \quad \int \mathcal{F}^{-1}(f)h = \int f\mathcal{F}^{-1}(h).$$

Now, for all $h \in S$,

$$\begin{aligned}\int \hat{f}(y)h(y)\,dy &= \int f(x)\hat{h}(x)\,dx \\ &= \int f(x)\left(\int h(y)e^{-i2\pi x\cdot y}\,dy\right)dx \\ &= \int \left(\int f(x)e^{-i2\pi x\cdot y}\,dx\right)h(y)\,dy \quad \text{(By Fubini's Theorem)}\end{aligned}$$

Thus,

$$\hat{f}(y) = \int f(x)e^{-i2\pi x\cdot y}\,dx \quad \text{a.e.}$$

Fubini holds since $\iint |f(x)|\cdot|h(y)|\cdot|e^{-i2\pi x\cdot y}|\,dx\,dy = |f|_{L^1}\cdot|h|_{L^1} < \infty$. An analogous proof holds for $\mathcal{F}^{-1}$.

Corollary 2

Let $f \in L^2$. Then,

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \lim_{n\to\infty}\int_{|x|\le n} f(x)e^{-i2\pi x\cdot\xi}\,dx \text{ in } L^2,$$

$$\mathcal{F}^{-1}f(x) = \lim_{n\to\infty}\int_{|\xi|\le n} f(\xi)e^{i2\pi x\cdot\xi}\,d\xi \text{ in } L^2.$$

Proof. Let $f \in L^2$. Set $f_n = \chi_{|x|\le n}f$. Then $f_n \in L^1\cap L^2$ (by Hölder, $\int|f_n| \le |f_n|_{L^2}\,|\chi_{|x|\le n}|_{L^2} < \infty$) and $f_n \to f$ in L^2 by the dominated convergence theorem, since $|f_n| \le |f|$ and $|f|^2$ is integrable. By Proposition 6, since $f_n \in L^1\cap L^2$,

$$\hat{f}_n(\xi) = \int_{\mathbb{R}^d} f_n(x)e^{-i2\pi x\cdot\xi}\,dx = \int_{|x|\le n} f(x)e^{-i2\pi x\cdot\xi}\,dx.$$

Since $\mathcal{F}$ is unitary on L^2, $|\hat{f}_n - \hat{f}|_{L^2} = |f_n - f|_{L^2} \to 0$, so $\hat{f}_n \to \hat{f}$ in L^2. Hence

$$\hat{f}(\xi) = \lim_{n\to\infty}\int_{|x|\le n} f(x)e^{-i2\pi x\cdot\xi}\,dx \quad \text{in } L^2.$$

An analogous argument with $g_n = \chi_{|\xi|\le n}f$ proves the formula for $\mathcal{F}^{-1}$.

Some Applications of the Fourier Transform

1. Heisenberg Principle

Let $\psi \in S(\mathbb{R})$ be a wave function, $\int_{\mathbb{R}}|\psi(x)|^2\,dx = 1$. Then, in quantum mechanics, $|\psi(x)|^2$ is the pdf of a particle's location and

$$\int |\hat{\psi}(\xi)|^2\,d\xi = \int |\psi(x)|^2\,dx = 1.$$

$|\hat{\psi}(\xi)|^2$ is the pdf of a particle's momentum. If x_0, ξ_0 are the measured location and momentum, then their mean square errors are

$$(\Delta x)^2 = \int_{\mathbb{R}} |x - x_0|^2 \, |\psi(x)|^2 \, dx, \qquad (\Delta \xi)^2 = \int_{\mathbb{R}} |\xi - \xi_0|^2 \, |\hat{\psi}(\xi)|^2 \, d\xi.$$

Theorem 2 (Heisenberg Uncertainty Principle). Let $\psi \in S(\mathbb{R})$ with $\int_{\mathbb{R}} |\psi(x)|^2 \, dx = 1$. Then

$$(\Delta x)^2 (\Delta \xi)^2 \geq \frac{1}{16\pi^2}.$$

Proof. First take $x_0 = \xi_0 = 0$, so that $(\Delta x)^2 = \int x^2 |\psi(x)|^2 \, dx$ and $(\Delta \xi)^2 = \int \xi^2 |\hat{\psi}(\xi)|^2 \, d\xi$. Since $\psi \in S(\mathbb{R})$, it decays faster than any polynomial, so $x|\psi(x)|^2 \to 0$ as $|x| \to \infty$. Integrating by parts,

$$\begin{aligned} 1 = \int_{-\infty}^{\infty} |\psi(x)|^2 \, dx &= \Big[x |\psi(x)|^2 \Big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} x \big(\psi'(x) \overline{\psi(x)} + \psi(x) \overline{\psi'(x)} \big) \, dx \\ &= -2 \operatorname{Re} \int_{-\infty}^{\infty} x \, \psi'(x) \, \overline{\psi(x)} \, dx. \end{aligned}$$

Hence, by the triangle inequality and then Cauchy–Schwarz,

$$1 \leq 2 \int_{-\infty}^{\infty} |x \, \psi(x)| \, |\psi'(x)| \, dx \leq 2 \left(\int x^2 |\psi(x)|^2 \, dx \right)^{1/2} \left(\int |\psi'(x)|^2 \, dx \right)^{1/2} = 2 \, (\Delta x) \left(\int |\psi'|^2 \, dx \right)^{1/2}.$$

By Plancherel's equality and Property 1, $\widehat{\psi'}(\xi) = i2\pi\xi \, \hat{\psi}(\xi)$, so

$$\int |\psi'(x)|^2 \, dx = \int |\widehat{\psi'}(\xi)|^2 \, d\xi = \int (2\pi|\xi|)^2 |\hat{\psi}(\xi)|^2 \, d\xi = 4\pi^2 (\Delta \xi)^2.$$

Therefore $1 \leq 2(\Delta x) \cdot 2\pi(\Delta \xi) = 4\pi(\Delta x)(\Delta \xi)$, which gives

$$(\Delta x)^2 (\Delta \xi)^2 \geq \frac{1}{16\pi^2}.$$

For general x_0, ξ_0, apply the result to $\varphi(x) = e^{-i2\pi\xi_0 x} \psi(x + x_0)$. Then $\int |\varphi|^2 = 1$, $|\hat{\varphi}(\xi)|^2 = |\hat{\psi}(\xi + \xi_0)|^2$, and a change of variables gives $\int x^2 |\varphi(x)|^2 dx = (\Delta x)^2$ and $\int \xi^2 |\hat{\varphi}(\xi)|^2 d\xi = (\Delta \xi)^2$. Applying the case above to φ yields the general inequality.

2. Helmholtz Equation in $S = S(\mathbb{R}^d)$

Consider

$$u(x) - \Delta u(x) = f(x), \quad x \in \mathbb{R}^d, \; f \in S, \tag{H}$$

where $\Delta u(x) = \frac{\partial^2}{\partial x_1^2} u(x) + \cdots + \frac{\partial^2}{\partial x_d^2} u(x)$. We found (Example 1, Property 1)

$$\widehat{\Delta u}(\xi) = -4\pi^2 |\xi|^2 \, \hat{u}(\xi).$$

Proposition 7

For any $f \in S$, there exists a unique $u \in S$ solving (H); i.e., $I - \Delta : S \to S$ is bijective.

Proof. Let $f \in S$. Note that $\hat{f} \in S$ and, since $1 + 4\pi^2|\xi|^2$ is a polynomial that is bounded below by 1 with $\frac{1}{1+4\pi^2|\xi|^2} \in C^\infty$ having all derivatives bounded, the product $\frac{1}{1+4\pi^2|\xi|^2}\,\hat{f}(\xi) \in S$.

(1) Uniqueness. Let $u \in S$ with $u - \Delta u = f$. Taking Fourier transforms,

$$\widehat{u - \Delta u}(\xi) = \hat{f}(\xi) \implies (1 + 4\pi^2|\xi|^2)\,\hat{u}(\xi) = \hat{f}(\xi) \implies \hat{u}(\xi) = \frac{\hat{f}(\xi)}{1 + 4\pi^2|\xi|^2}.$$

Thus $\hat{u}$ is uniquely determined, and since $\mathcal{F} : S \to S$ is a bijection, u is unique.

(2) Existence. Let $f \in S$ and set $g(\xi) = \dfrac{\hat{f}(\xi)}{1 + 4\pi^2|\xi|^2} \in S$. Then $u(x) = (\mathcal{F}^{-1}g)(x) \in S$ solves (H): indeed $\hat{u} = g$, so

$$\widehat{u - \Delta u}(\xi) = (1 + 4\pi^2|\xi|^2)\,\hat{u}(\xi) = (1 + 4\pi^2|\xi|^2)\,g(\xi) = \hat{f}(\xi),$$

and injectivity of $\mathcal{F}$ gives $u - \Delta u = f$.

Tempered Distributions

Notation. Let $S' = S'(\mathbb{R}^d)$ be the space of all linear continuous maps $T : S \to \mathbb{C}$.

Recall. A linear map $T : S \to \mathbb{C}$ is continuous if and only if there exist $m \geq 1$, $C \geq 0$, $\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{N}_0^d$, and $N_1, N_2, \ldots, N_m \in \mathbb{N}_0$ such that

$$|T(f)| \leq C \sum_{k=1}^{m} |f|_{N_k,\alpha_k}, \quad f \in S.$$

Equivalently, a linear map $T : S \to \mathbb{C}$ is continuous if and only if there exist $N \in \mathbb{N}_0$ and $C \geq 0$ such that $|T(f)| \leq C|f|_{(N)}$.

Definition. $T \in S'$ is called a *tempered distribution*, and $S' = S'(\mathbb{R}^d)$ is called the space of tempered distributions.

Example 1 ($\mathcal{O}_M$)

Let $\mathcal{O}_M$ be the space of all $f \in C^\infty$ such that for every $\alpha \in \mathbb{N}_0^d$ there exist $C = C(\alpha) \geq 0$ and $N = N(\alpha) \in \mathbb{N}_0$ with

$$|\partial^\alpha f(x)| \leq C(1 + |x|)^N, \quad x \in \mathbb{R}^d.$$

(a) Every $g \in \mathcal{O}_M$ defines $T_g \in S'$ by $T_g(f) = \displaystyle\int_{\mathbb{R}^d} g(x)f(x)\,dx$, $f \in S$.

Proof. Suppose $|g(x)| \le C(0)(1+|x|)^N$ for $x \in \mathbb{R}^d$ (the bound for $\alpha = 0$). Then

$$\begin{aligned}\int |g(x)| \cdot |f(x)|\, dx &\le C(0) \int (1+|x|)^N |f(x)|\, dx \\ &= C(0) \int (1+|x|)^{N+d+1} |f(x)| \cdot \frac{dx}{(1+|x|)^{d+1}} \\ &\le C(0)\, |f|_{N+d+1,\,0} \int \frac{dx}{(1+|x|)^{d+1}}.\end{aligned}$$

Hence

$$|T_g(f)| \le \left| \int g(x) f(x)\, dx \right| \le C\, |f|_{N+d+1,\,0}, \qquad C = C(0) \int \frac{dx}{(1+|x|)^{d+1}} < \infty,$$

so T_g is continuous, i.e. $T_g \in S'$.

(b) The map $\mathcal{O}_M \ni g \mapsto T_g \in S'$ is injective.

Proof. Let $g_1, g_2 \in \mathcal{O}_M$ with $T_{g_1}(f) = T_{g_2}(f)$ for all $f \in S$. Then $\int g_1(x) f(x)\, dx = \int g_2(x) f(x)\, dx$ for all $f \in S$. Since g_1, g_2 are locally integrable and $\int g_1 f = \int g_2 f$ for all $f \in C_c^\infty$, it follows that $g_1 = g_2$ a.e.

(c) Let $\alpha \in \mathbb{N}_0^d$. Then $\partial^\alpha g \in \mathcal{O}_M$ for $g \in \mathcal{O}_M$, and since $f \in S$ decays rapidly, integration by parts gives

$$\langle \partial^\alpha g, f \rangle = \int_{\mathbb{R}^d} \partial^\alpha g \cdot f\, dx = (-1)^{|\alpha|} \int g\, \partial^\alpha f\, dx = (-1)^{|\alpha|} \langle g, \partial^\alpha f \rangle = (-1)^{|\alpha|} T_g(\partial^\alpha f), \quad f \in S.$$

Notation. $C_c^\infty \subseteq S \subseteq \mathcal{O}_M \subseteq S'$. We also write $T_g = g$ and $T_g(f) = \langle g, f \rangle$ for $f \in S$.

Definition. Let $T_n, T \in S'$. We say $T_n \to T$ in S' if $T_n(f) \to T(f)$ for every $f \in S$.

Example 2 (Polynomial Growth)

We say a measurable function $g : \mathbb{R}^d \to \mathbb{C}$ is of *polynomial growth* if

$$|g(x)| \le C(1+|x|)^N, \quad x \in \mathbb{R}^d, \text{ for some } C > 0,\ N \ge 1.$$

Remark. If g is of polynomial growth, then g defines $T_g \in S'$ by $T_g(f) = \int gf$, $f \in S$ (equivalently $\langle g, f \rangle = \int gf$), and $|T_g(f)| \le C|f|_{N+d+1,\,0}$, $f \in S$. Again we write $T_g = g$.

For $\alpha \in \mathbb{N}_0^d$ we define the distributional derivative $\partial^\alpha g \in S'$ by

$$\langle \partial^\alpha g, f \rangle = (-1)^{|\alpha|} \langle g, \partial^\alpha f \rangle = (-1)^{|\alpha|} T_g(\partial^\alpha f), \quad f \in S. \tag{\#}$$

Formally this reads $\langle \partial^\alpha g, f \rangle = \int \partial^\alpha g(x) f(x)\, dx = (-1)^{|\alpha|} \int g(x) \partial^\alpha f(x)\, dx$. As an operator,

$$\partial^\alpha T_g = (-1)^{|\alpha|} T_g \circ \partial^\alpha, \qquad S \xrightarrow{\partial^\alpha} S \xrightarrow{T_g} \mathbb{C}.$$

More generally, (#) defines $\partial^\alpha T$ for any $T \in S'$ by $(\partial^\alpha T)(f) = (-1)^{|\alpha|} T(\partial^\alpha f)$, $f \in S$.

Definition. $T \in S'$ is *regular* if $T = T_g$ where g is of polynomial growth.

E.g. Let

$$g(x) = \begin{cases} x, & x \geq 0 \\ 0, & x < 0 \end{cases} \qquad \text{(the ramp function).}$$

(a) Find $\frac{d}{dx} g = g'$. For $f \in S$,

$$\begin{aligned} \left\langle \tfrac{d}{dx} g, f \right\rangle &= -\int_{-\infty}^{\infty} g(x) f'(x)\, dx = -\int_{0}^{\infty} x f'(x)\, dx \\ &= -\Big[x f(x) \Big]_0^{\infty} + \int_0^{\infty} f(x)\, dx = \int_0^{\infty} f(x)\, dx = \int_{-\infty}^{\infty} H(x) f(x)\, dx, \end{aligned}$$

where the boundary term vanishes since $f \in S$. Hence $g' = H$, the Heaviside function,

$$H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}, \qquad H = g' = \frac{d}{dx} g.$$

(b) Find $\frac{dH}{dx}$. For $f \in S$,

$$\begin{aligned} \left\langle \tfrac{dH}{dx}, f \right\rangle &= -\int_{-\infty}^{\infty} H(x) f'(x)\, dx \\ &= -\int_0^{\infty} f'(x)\, dx \\ &= -\Big[f(x) \Big]_0^{\infty} = f(0) \\ &= \int_{-\infty}^{\infty} f(x)\, \delta_0(dx) = \int_{-\infty}^{\infty} f(x)\, dH(x). \end{aligned}$$

Thus $\frac{dH}{dx} = \delta_0$, the Dirac measure, where (informally)

$$\delta_0(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases} \qquad \text{(Dirac measure),} \qquad \langle \delta_0, f \rangle = f(0).$$

Formally, $\displaystyle\int_{-\infty}^{\infty} f(x) H'(x)\, dx = \int_{-\infty}^{\infty} f(x)\, dH(x).$

Example 3 (L^p)

Let $g \in L^p$ with $p \geq 1$. Then g defines $T_g \in S'$ by $T_g(f) = \displaystyle\int g(x) f(x)\, dx$, $f \in S$.

Proof. By Hölder's inequality, with $\frac{1}{p} + \frac{1}{q} = 1$,

$$\int |g(x)| \cdot |f(x)|\, dx \leq |g|_{L^p}\, |f|_{L^q}.$$

Now, for $f \in S$, writing $m = \lceil \frac{d+1}{q} \rceil$,

$$|f|_{L^q}^q = \int |f(x)|^q \, dx = \int \left(|f(x)|(1+|x|)^{\frac{d+1}{q}} \right)^q \cdot \frac{dx}{(1+|x|)^{d+1}} \leq |f|_{m,0}^q \int \frac{dx}{(1+|x|)^{d+1}},$$

where $\int \frac{dx}{(1+|x|)^{d+1}} = C < \infty$. Then $|f|_{L^q} \leq C^{1/q} |f|_{m,0}$, so

$$|T_g(f)| \leq |g|_{L^p} \, |f|_{L^q} \leq C^{1/q} \, |g|_{L^p} \, |f|_{m,0} < \infty,$$

showing $T_g \in S'$. The map $g \in L^p \mapsto T_g \in S'$ is injective (similar proof to $\mathcal{O}_M$ for injectivity).

Tempered Distributions (continued)

We define $L^p \subseteq S'$ and often write $g = T_g$, where

$$T_g(f) = \langle g, f \rangle = \int g(x) f(x) \, dx, \quad f \in S.$$

We define $\partial^\alpha g$ by

$$\langle \partial^\alpha g, f \rangle = (-1)^{|\alpha|} \langle g, \partial^\alpha f \rangle, \quad f \in S.$$

In particular $L^2 \subseteq S'$. Recall that if $g \in L^2$, then $\mathcal{F} g = \hat{g}$ and $\mathcal{F}^{-1} g \in L^2$. By Proposition 3 of Chapter 7,

$$\int \hat{g} \, f = \int g \, \hat{f}, \quad \int \mathcal{F}^{-1} g \, f = \int g \, \mathcal{F}^{-1} f, \quad f \in S,$$

and this even holds for $f \in L^2$:

$$\langle \hat{g}, f \rangle = \langle g, \hat{f} \rangle, \quad \langle \mathcal{F}^{-1} g, f \rangle = \langle g, \mathcal{F}^{-1} f \rangle, \quad f \in S.$$

For $g \in L^p$ with $p \geq 1$, we define $\mathcal{F} g = \hat{g}$, $\mathcal{F}^{-1} g \in S'$ by

$$\langle \mathcal{F} g, f \rangle = \langle \hat{g}, f \rangle = \langle g, \mathcal{F} f \rangle = \langle g, \hat{f} \rangle, \quad \langle \mathcal{F}^{-1} g, f \rangle = \langle g, \mathcal{F}^{-1} f \rangle, \quad f \in S.$$

Note. $S \xrightarrow{\mathcal{F}} S \xrightarrow{T_g} \mathbb{C}$, i.e. $\mathcal{F} g = T_g \circ \mathcal{F}$.

Example 4 ($\mathcal{M}_{\text{mod}}$)

Let $\mathcal{M}_{\text{mod}}$ (moderate) be the space of signed Borel measures μ on $\mathbb{R}^d$ of *moderate growth*:

$$\int (1+|x|)^{-N} \, d|\mu| < \infty \quad \text{for some integer } N \geq 1.. \tag{\#}$$

E.g. Lebesgue measure on $\mathbb{R}^d$: $N = d+1$ works (indeed any $N = d + \epsilon$, $\epsilon > 0$).

Every $\mu \in \mathcal{M}_{\text{mod}}$ defines $T_\mu \in S'$ by $T_\mu(f) = \int f \, d\mu$, $f \in S$.

Proof (continuity).

$$\int |f| \, d|\mu| = \int |f(x)|(1+|x|)^N \cdot \frac{1}{(1+|x|)^N} \, d|\mu| \leq |f|_{N,0} \int \frac{d|\mu|}{(1+|x|)^N} < \infty \quad \text{by (\#)},$$

for $f \in S$. Hence $|T_\mu(f)| \le C\,|f|_{N,0}$, so $T_\mu \in S'$.

The map $\mu \in \mathcal{M}_{\text{mod}} \mapsto T_\mu \in S'$ is injective; we write $\mathcal{M}_{\text{mod}} \subseteq S'$, $T_\mu(f) = \langle \mu, f\rangle$ for $f \in S$, and $\mu = T_\mu$. We define

$$\langle \partial^\alpha \mu, f\rangle = (-1)^{|\alpha|}\langle \mu, \partial^\alpha f\rangle, \quad \langle \mathcal{F}\mu, f\rangle = \langle \mu, \mathcal{F}f\rangle, \quad \langle \mathcal{F}^{-1}\mu, f\rangle = \langle \mu, \mathcal{F}^{-1}f\rangle, \quad f \in S.$$

Let μ be a finite signed Borel measure on $\mathbb{R}^d$. Then, by Fubini,

$$\begin{aligned}\langle \mathcal{F}^{-1}\mu, f\rangle = \langle \mu, \mathcal{F}^{-1}f\rangle = \int \mathcal{F}^{-1}f\,d\mu = \int \Big(\int_{\mathbb{R}^d} f(\xi)e^{i2\pi x\cdot\xi}\,d\xi\Big)\,d\mu(x)\\ = \int_{\mathbb{R}^d} f(\xi)\Big(\int e^{i2\pi x\cdot\xi}\,d\mu(x)\Big)\,d\xi = \int_{\mathbb{R}^d} f(\xi)\,\phi(\xi)\,d\xi,\end{aligned}$$

where

$$\phi(\xi) = \int e^{i2\pi x\cdot\xi}\,d\mu(x) = E\big[e^{i2\pi x\cdot\xi}\big], \quad \xi \in \mathbb{R}^d,$$

(the characteristic function of μ). Thus $\mathcal{F}^{-1}\mu = \phi$ in S', where $\phi = T_\phi$ is defined by $T_\phi(f) = \int f\phi$, $f \in S$.

Main Operations on $S' = S'(\mathbb{R}^d)$

Differentiation. Given $T \in S'$ and $\alpha \in \mathbb{N}_0^d$, we define $\partial^\alpha T \in S'$ by

$$\partial^\alpha T(f) = (-1)^{|\alpha|}T(\partial^\alpha f), \quad f \in S, \quad \text{i.e.} \quad \langle \partial^\alpha T, f\rangle = (-1)^{|\alpha|}\langle T, \partial^\alpha f\rangle, \quad f \in S,$$

where, for $R \in S'$, $R(f) = \langle R, f\rangle$.

Fourier Transform. Given $T \in S'$, we define $\mathcal{F}T = \hat{T}$, $\mathcal{F}^{-1}T \in S'$ by $\hat{T}(f) = T(\hat{f})$, i.e.

$$\mathcal{F}T(f) = T(\mathcal{F}f), \quad \mathcal{F}^{-1}T(f) = T(\mathcal{F}^{-1}f), \quad f \in S.$$

Notation. $\langle \hat{T}, f\rangle = \langle T, \hat{f}\rangle$, $\quad \langle \mathcal{F}^{-1}T, f\rangle = \langle T, \mathcal{F}^{-1}f\rangle$.

Multiplication by $F \in \mathcal{O}_M$***.*** Given $T \in S'$ and $F \in \mathcal{O}_M$, we define $FT \in S'$ by

$$FT(f) = T(Ff), \quad f \in S, \quad \text{i.e.} \quad \langle FT, f\rangle = \langle T, Ff\rangle, \quad f \in S,$$

which is well defined since $f \in S \mapsto Ff \in S$ is continuous.

Proposition 1

$\mathcal{F}, \mathcal{F}^{-1} : S' \to S'$ are bijections (recall $S \subseteq L^2 \subseteq S'$).

Proof. Let $f \in S$. Then

$\mathcal{F}^{-1}\mathcal{F}T(f) = \mathcal{F}T(\mathcal{F}^{-1}f) = T(\mathcal{F}\mathcal{F}^{-1}f) = T(f)$, $\mathcal{F}\mathcal{F}^{-1}T(f) = \mathcal{F}^{-1}T(\mathcal{F}f) = T(\mathcal{F}^{-1}\mathcal{F}f) = T(f)$.

Thus $\mathcal{F}^{-1}\mathcal{F}T = T = \mathcal{F}\mathcal{F}^{-1}T$.

Example

Let $T \in S'$, $\alpha \in \mathbb{N}_0^d$. Show that $\widehat{\partial^\alpha T} = (i2\pi\xi)^\alpha \hat{T}$.

We know $(i2\pi\xi)^\alpha \in \mathcal{O}_M$. Let $f \in S$. Then

$$\begin{aligned}\widehat{\partial^\alpha T}(f) &= \partial^\alpha T(\hat{f}) = (-1)^{|\alpha|} T(\partial^\alpha \hat{f}) = (-1)^{|\alpha|} T(\hat{g}) \quad \text{with } g(x) = (-i2\pi x)^\alpha f(x) \in S \\ &= (-1)^{|\alpha|} T\big(\widehat{(-i2\pi\cdot)^\alpha f}\big) = (-1)^{|\alpha|} \hat{T}\big((-i2\pi\cdot)^\alpha f\big) \quad \text{with } (-i2\pi\cdot)^\alpha \in \mathcal{O}_M \\ &= (-1)^{|\alpha|}(-i2\pi\cdot)^\alpha \hat{T}(f) = (i2\pi\cdot)^\alpha \hat{T}(f),\end{aligned}$$

using Property 3, $\partial^\alpha \hat{f} = \hat{g}$ with $g = (-i2\pi\cdot)^\alpha f$. Hence $\widehat{\partial^\alpha T} = (i2\pi\xi)^\alpha \hat{T}$.

Recall. Let $g(x) = \begin{cases} x, & x \geq 0 \\ 0, & x < 0 \end{cases}$, $g \in S'$. We found $\frac{d^2}{dx^2} g = \delta_0$, and $\delta_0 \in S'$ with $\int_{\mathbb{R}^d} f(x)\, d\delta_0(x) = f(0)$. Also recall that $g : \mathbb{R}^d \to \mathbb{C}$ is of *polynomial growth* if $|g(x)| \leq C(1+|x|)^N$, $x \in \mathbb{R}^d$, for some $C > 0$ and integer $N \geq 1$.

Theorem 1 (Structure Theorem)

For each $T \in S'$, there exist a continuous function g of polynomial growth and $\alpha \in \mathbb{N}_0^d$ such that $T = \partial^\alpha g$, i.e.

$$T(f) = \langle T, f\rangle = \langle \partial^\alpha g, f\rangle = (-1)^{|\alpha|}\langle g, \partial^\alpha f\rangle = (-1)^{|\alpha|}\int_{\mathbb{R}^d} g(x)\, \partial^\alpha f(x)\, dx.$$

Proof. We prove it for $d = 1$. In particular, we prove the following first.

Lemma 1

Let $h : \mathbb{R} \to \mathbb{C}$ be of polynomial growth. Then there exists a continuous function H of polynomial growth so that

$$\langle h, f^{(n)}\rangle = \langle H, f^{(n+1)}\rangle, \quad f \in S,\ n \in \mathbb{N}_0, \quad \text{i.e.} \quad \int_{-\infty}^{\infty} h(x) f^{(n)}(x)\, dx = \int_{-\infty}^{\infty} H(x) f^{(n+1)}(x)\, dx.$$

Proof. Set $G(x) = \int_0^x h(t)\, dt$, $x \in \mathbb{R}$. Suppose $|h(x)| \leq C_1(1+|x|)^N$. Then, using $|x| \leq \frac{|x|^2+1}{2}$,

$$|h(x)| \leq C_2(1+|x|^2)^N, \quad \text{and also} \quad |h(x)| \leq C_3(1+|x|^{2N}), \quad x \in \mathbb{R},$$

(since $|x|^k \leq \frac{|x|^{2N}}{C_4} + C_5$). Then G is a continuous function of polynomial growth: for $x \geq 0$,

$$|G(x)| \leq \int_0^x |h(t)|\, dt \leq C_6 \int_0^x (1+t^{2N})\, dt = C_6 x + C_6 \frac{x^{2N+1}}{2N+1},$$

and for $x < 0$, $|G(x)| \le \int_x^0 |h(t)|\,dt \le C_6|x| + C_6\frac{|x|^{2N+1}}{2N+1}$; hence $|G(x)| \le C_7(1+|x|)^{2N+1}$.

$$\begin{aligned}\langle G, f^{(n+1)}\rangle &= \int_{-\infty}^{\infty} G(x) f^{(n+1)}(x)\,dx \\ &= \Big[G(x)f^{(n)}(x)\Big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f^{(n)}(x)h(x)\,dx \\ &= 0 - \langle h, f^{(n)}\rangle \quad (f^{(n)} \in S,\ h \in S').\end{aligned}$$

Take $H = -G$, which finishes the proof of the lemma.

Now we return to Theorem 1. Let $T \in S'$. There exist $C > 0$, $N \in \mathbb{N}_0$ such that

$$|T(f)| \le C|f|_{(N)} \le C\sum_{k=0}^{N}\sup_x (1+|x|)^N|f^{(k)}(x)| \le C_1\sum_{k=0}^{N}\sup_x(1+|x|^2)^N|f^{(k)}(x)|, \quad f \in S. \tag{1}$$

Let $\varphi(x) = (1+|x|^2)^{-N}$. Note $\varphi \in \mathcal{O}_M$ and $|\varphi^{(\ell)}(x)| \le C_\ell\,\varphi(x)$ for all $\ell \in \mathbb{N}_0$, $x \in \mathbb{R}$. Consider $\varphi T(f) = T(\varphi f)$, $f \in S$.

Claim 1. $|\varphi T(f)| \le C_2 \sum_{k=0}^{N}\sup_x |f^{(k)}(x)|$, $f \in S$.

Proof. By (1), $|\varphi T(f)| = |T(\varphi f)| \le C_1\sum_{k=0}^{N}\sup_x(1+|x|^2)^N|(\varphi f)^{(k)}(x)|$. By Leibniz,

$$|(\varphi f)^{(k)}(x)| \le \sum_{j=0}^{k}\frac{k!}{j!(k-j)!}|f^{(j)}(x)|\,|\varphi^{(k-j)}(x)| \le \sum_{j=0}^{k} C'_{k,j}\,|f^{(j)}(x)|\,\varphi(x), \quad C'_{k,j} = \tbinom{k}{j}C_{k-j}.$$

Since $(1+|x|^2)^N\varphi(x) = 1$,

$$|\varphi T(f)| \le C_1\sum_{k=0}^{N}\sum_{j=0}^{k} C'_{k,j}\sup_x |f^{(j)}(x)| \le C_2\sum_{k=0}^{N}\sup_x |f^{(k)}(x)|.$$

Claim 2. There exist complex (finite Borel) measures μ_k, $k = 0, 1, \ldots, N$, on $\mathbb{R}$ such that

$$\varphi T(f) = \sum_{k=0}^{N}\int_{\mathbb{R}} f^{(k)}(x)\,d\mu_k, \quad f \in S.$$

Proof. Let $C_0(\mathbb{R})$ be the Banach space of continuous functions on $\mathbb{R}$ vanishing at $\pm\infty$, with norm $|g|_u = \sup_x |g(x)|$. Let $E = \underbrace{C_0(\mathbb{R}) \times \cdots \times C_0(\mathbb{R})}_{N+1}$ with norm $|(g_0, \ldots, g_N)|_u = |g_0|_u + \cdots + |g_N|_u$; E is a Banach space. Let

$$M = \Big\{(f^{(0)}, f^{(1)}, \ldots, f^{(N)}) : f \in S\Big\} \subseteq E,$$

and consider $\Lambda : M \to \mathbb{C}$ defined by $\Lambda(f^{(0)}, f^{(1)}, \ldots, f^{(N)}) = \varphi T(f)$. By Claim 1,

$$|\Lambda(f^{(0)}, \ldots, f^{(N)})| = |\varphi T(f)| \le C_2 \, |(f^{(0)}, \ldots, f^{(N)})|_u,$$

so Λ is bounded on M. By a corollary of the Hahn–Banach theorem, Λ extends to $\Lambda \in E^*$ with $\|\Lambda\|_{E^*} \le C_2$. Now $|\Lambda(g_0, 0, \ldots, 0)| \le C_2 |g_0|_u$ for $g_0 \in C_0(\mathbb{R})$, so by the Riesz representation theorem there exists a finite complex Borel measure μ_0 on $\mathbb{R}$ with $\Lambda(g_0, 0, \ldots, 0) = \int_{\mathbb{R}} g_0 \, d\mu_0$. Continuing slot by slot, we obtain $\mu_1, \ldots, \mu_N$ with

$$\Lambda(g_0, g_1, \ldots, g_N) = \sum_{k=0}^{N} \int_{\mathbb{R}} g_k(x) \, d\mu_k, \quad (g_0, \ldots, g_N) \in E$$

(for example, $\Lambda(g_0, g_1) = \Lambda(g_0, 0) + \Lambda(0, g_1)$). Hence $\varphi T(f) = \Lambda(f^{(0)}, \ldots, f^{(N)}) = \sum_{k=0}^{N} \int_{\mathbb{R}} f^{(k)} \, d\mu_k$.

Claim 3. There exist bounded measurable functions F_k, $k = 0, 1, \ldots, N$, such that

$$\varphi T(f) = \sum_{k=0}^{N} \int_{\mathbb{R}} f^{(k+1)}(x) \, F_k(x) \, dx, \quad f \in S.$$

Proof. Let $G_k(x) = \mu_k\big((-\infty, x]\big)$, $x \in \mathbb{R}$; then G_k is bounded (as μ_k is finite) and of bounded variation. Then

$$\begin{aligned}\varphi T(f) &= \sum_{k=0}^{N} \int_{-\infty}^{\infty} f^{(k)}(x) \, d\mu_k(x) = \sum_{k=0}^{N} \int_{-\infty}^{\infty} f^{(k)}(x) \, dG_k(x) \\ &= \sum_{k=0}^{N} \Big[f^{(k)} G_k \Big]_{-\infty}^{\infty} - \sum_{k=0}^{N} \int_{-\infty}^{\infty} f^{(k+1)}(x) G_k(x) \, dx.\end{aligned}$$

The boundary terms vanish ($f^{(k)} \in S$, G_k bounded). Set $F_k = -G_k$.

Finishing the proof. Since $\frac{1}{\varphi} = (1 + |x|^2)^N \in \mathcal{O}_M$ and $\frac{1}{\varphi} f \in S$,

$$T(f) = \frac{1}{\varphi} \cdot \varphi T(f) = \varphi T\Big(\tfrac{1}{\varphi} f\Big) = \sum_{k=0}^{N} \int_{\mathbb{R}} F_k(x) \left(\tfrac{1}{\varphi} f\right)^{(k+1)} (x) \, dx.$$

By Leibniz, $\left(\frac{1}{\varphi} f\right)^{(k+1)} = \sum_{\ell=0}^{k+1} \binom{k+1}{\ell} f^{(\ell)} \left(\frac{1}{\varphi}\right)^{(k+1-\ell)}$, so

$$T(f) = \sum_{k=0}^{N} \sum_{\ell=0}^{k+1} \int_{\mathbb{R}} f^{(\ell)}(x) \, H_{k,\ell}(x) \, dx, \quad H_{k,\ell} = \binom{k+1}{\ell} F_k \left(\tfrac{1}{\varphi}\right)^{(k+1-\ell)},$$

where each $H_{k,\ell}$ is measurable of polynomial growth (bounded $\times$ polynomial growth). Applying Lemma 1 repeatedly to raise every term to order $N+2$,

$$T(f) = \sum_{k=0}^{N} \sum_{\ell=0}^{k+1} \int_{\mathbb{R}} f^{(N+2)}(x) \, \widetilde{H}_{k,\ell}(x) \, dx,$$

where the $\widetilde{H}_{k,\ell}$ are continuous of polynomial growth. Set

$$g(x) = (-1)^{N+2} \sum_{k=0}^{N} \sum_{\ell=0}^{k+1} \widetilde{H}_{k,\ell}(x), \quad x \in \mathbb{R},$$

a continuous function of polynomial growth. Then $T(f) = (-1)^{N+2} \int g(x) f^{(N+2)}(x)\, dx = \langle \partial^{N+2} g, f\rangle$, i.e. $T = \partial^{N+2} g$ (so $\alpha = N+2$). This concludes the proof of Theorem 1.

Some Applications of Tempered Distributions

Helmholtz Equation in $S' = S'(\mathbb{R}^d)$

Consider

$$u - \Delta u = g \ \text{ in } S'. \tag{H}$$

Given $g \in S'$, find $u \in S'$ solving (H).

Proposition 2

For any $g \in S'$, there exists a unique $u \in S'$ solving (H); i.e. $I - \Delta : S' \to S'$ is bijective. Moreover

$$u = \mathcal{F}^{-1}\Big(\frac{1}{1 + 4\pi^2|\xi|^2}\, \hat{g}\Big), \quad \text{where } \frac{1}{1+4\pi^2|\xi|^2} \in \mathcal{O}_M.$$

Proof. Note that both $1 + 4\pi^2|\xi|^2$ and $\frac{1}{1+4\pi^2|\xi|^2}$ are in $\mathcal{O}_M$. Using $\widehat{\partial^\alpha T} = (i2\pi\xi)^\alpha \hat{T}$ (so $\widehat{\Delta u} = -4\pi^2|\xi|^2\hat{u}$),

$$\begin{aligned} u - \Delta u = g \text{ in } S' \iff \widehat{u - \Delta u} = \hat{g} \text{ in } S' &\iff (1 + 4\pi^2|\xi|^2)\hat{u} = \hat{g} \text{ in } S' \\ \iff \hat{u} = \frac{1}{1+4\pi^2|\xi|^2}\hat{g} \text{ in } S' &\iff u = \mathcal{F}^{-1}\Big(\frac{1}{1+4\pi^2|\xi|^2}\hat{g}\Big) \text{ in } S'. \end{aligned}$$

This determines u uniquely and provides a solution, so $I - \Delta : S' \to S'$ is bijective.

(H) with $g \in L^2$

Definition. $H_2^2 = \big\{u \in S' : u,\, \partial^\alpha u \in L^2 \text{ for all } |\alpha| \le 2\big\}$ with norm

$$|u|_{2,2} = \Bigg(\sum_{|\alpha|\le 2} |\partial^\alpha u|_{L^2}^2\Bigg)^{1/2}.$$

In fact H_2^2 is a Hilbert space with inner product $\langle u, v\rangle = \sum_{|\alpha|\le 2} \int_{\mathbb{R}^d} \partial^\alpha u\, \overline{\partial^\alpha v}\, dx$.

Note. $u \in H_2^2$ means $u \in L^2$ and there exists $\partial^\alpha u \in L^2$ so that $\int \partial^\alpha u\, f\, dx = (-1)^{|\alpha|} \int u\, \partial^\alpha f\, dx$, $f \in S$, for all $|\alpha| \le 2$.

Proposition 3

For all $g \in L^2$, there exists a unique $u \in H_2^2$ solving (H). Moreover, $u - \Delta u = g$ a.e., and $|u|_{2,2} \leq C|g|_{L^2}$ with C independent of g. Also $I - \Delta : H_2^2 \to L^2$ is bijective.

Proof. Let $g \in L^2$ and $u = \mathcal{F}^{-1}\Big(\frac{1}{1+4\pi^2|\xi|^2}\hat{g}\Big) \in L^2$. Then for all $|\alpha| \leq 2$,

$$\widehat{\partial^\alpha u} = (i2\pi\xi)^\alpha \hat{u} = \frac{(i2\pi\xi)^\alpha}{1+4\pi^2|\xi|^2}\hat{g} \implies |\widehat{\partial^\alpha u}(\xi)| \leq |\hat{g}(\xi)|,$$

since for $|\alpha| \leq 2$, $|(i2\pi\xi)^\alpha| \leq (2\pi|\xi|)^{|\alpha|} \leq 1 + 4\pi^2|\xi|^2$. Hence $\partial^\alpha u \in L^2$ and, by Plancherel,

$$|\partial^\alpha u|_{L^2} = |\widehat{\partial^\alpha u}|_{L^2} \leq |\hat{g}|_{L^2} = |g|_{L^2}.$$

Summing over $|\alpha| \leq 2$ gives $|u|_{2,2} \leq C|g|_{L^2}$. Uniqueness follows from Proposition 2 (uniqueness in S'). Hence $I - \Delta : H_2^2 \to L^2$ is bijective.

Solution of (H) as a limit with $g \in L^2$***.*** Let $g \in L^2$. There exist $g_n \in S$ with $g_n \to g$ in L^2. For each $g_n \in S$ there is a unique $u_n \in S$ solving (H) (since $I - \Delta : S \to S$ is bijective, $S \subseteq L^2$, and S is dense in L^2). Then $(u_n - u_m) - \Delta(u_n - u_m) = g_n - g_m$, and by Proposition 3, for all $|\alpha| \leq 2$,

$$|\partial^\alpha u_n - \partial^\alpha u_m|_{L^2} \leq C|g_n - g_m|_{L^2} \to 0 \quad \text{as } n, m \to \infty.$$

Hence there exist $u_\alpha \in L^2$ with $\partial^\alpha u_n \to u_\alpha$ in L^2 for $|\alpha| \leq 2$; write $u = u_0$. For any $f \in S$ and $|\alpha| \leq 2$, $\int \partial^\alpha u_n\, f = (-1)^{|\alpha|} \int u_n\, \partial^\alpha f$, and passing to the limit,

$$\int u_\alpha\, f = (-1)^{|\alpha|} \int u\, \partial^\alpha f, \quad f \in S,$$

which means $\partial^\alpha u = u_\alpha$ in S'. Thus $u \in H_2^2$ and $u - \Delta u = g$ a.e.

(H) with $g \in H_2^s$

Definition. Let $s \in \mathbb{N}_0$. Define

$$H_2^s = \big\{u \in S' : u,\, \partial^\alpha u \in L^2,\ |\alpha| \leq s\big\}, \quad |u|_{s,2} = \left(\sum_{|\alpha|\leq s} |\partial^\alpha u|_{L^2}^2\right)^{1/2}.$$

Note. $H_2^s \subseteq L^2$.

Proposition 4

For any $g \in H_2^s$, there exists a unique $u \in H_2^{s+2}$ solving (H); i.e. $I - \Delta : H_2^{s+2} \to H_2^s$ is bijective.

Proof. $u - \Delta u = g$ implies $\partial^\alpha u - \Delta \partial^\alpha u = \partial^\alpha g$ in S'. Since $\partial^\alpha g \in L^2$ for all $|\alpha| \leq s$, by Proposition 3 we have, for all $|\beta| \leq 2$,

$$|\partial^{\beta+\alpha} u|_{L^2} = |\partial^\beta(\partial^\alpha u)|_{L^2} \leq C|\partial^\alpha g|_{L^2}.$$

Every γ with $|\gamma| \le s+2$ can be written $\gamma = \beta + \alpha$ with $|\alpha| \le s$, $|\beta| \le 2$, so

$$|\partial^\gamma u|_{L^2} \le C|g|_{s,2} \quad \text{for all } |\gamma| \le s+2.$$

Hence $u \in H_2^{s+2}$ solves (H), and uniqueness follows from Proposition 2. Therefore $I - \Delta : H_2^{s+2} \to H_2^s$ is bijective.

Weak Convergence

Let E be a normed vector space and $E^* = \mathcal{L}(E, \mathbb{R})$ be its dual. Let $x \in E$ and let $x_n \in E$ be a sequence "approximating" x. In what sense do we mean "approximating"? By the Hahn–Banach theorem,

$$|x_n - x|_E = \sup_{|\ell|_{E^*} \le 1} |\ell(x_n - x)| \to 0 \text{ as } n \to \infty,$$

if and only if for any $R > 0$, $\sup_{|\ell|_{E^*} \le R} |\ell(x_n) - \ell(x)| \to 0$ as $n \to \infty$. That is, norm convergence asks for $\ell(x_n) \to \ell(x)$ *uniformly* over bounded sets of functionals. Weak convergence relaxes this to pointwise.

Definition. We say $x_n \to x$ *weakly*, written $x_n \xrightarrow{w} x$, if

$$\ell(x_n) \to \ell(x) \text{ as } n \to \infty \text{ for all } \ell \in E^*, \quad \text{i.e.} \quad \langle \ell, x_n \rangle \to \langle \ell, x \rangle \text{ as } n \to \infty \text{ for all } \ell \in E^*.$$

Remark 1.

(a) Let $E = L^p$, $p \in [1, \infty)$. Then $(L^p)^* = L^q$ with $\frac{1}{p} + \frac{1}{q} = 1$. Therefore $f_n \xrightarrow{w} f$ in L^p if and only if $\int f_n g \to \int fg$ for all $g \in L^q$.

(b) Let X be a compact metric space and $E = C(X)$ with the $|f|_u$ (sup) norm. Then $[C(X)]^* = M(X)$, the space of finite signed Borel measures, and $f_n \xrightarrow{w} f$ if and only if $\int f_n \, d\mu \to \int f \, d\mu$ for all $\mu \in M(X)$.

(c) Let $E = H$ be a Hilbert space. Then $H^* = H$ and $x_n \xrightarrow{w} x$ if and only if $\langle z, x_n \rangle \to \langle z, x \rangle$ as $n \to \infty$ for all $z \in H$. (Recall Problem 3 of HW4: $x_n \to x$ in H if and only if $x_n \xrightarrow{w} x$ and $\limsup_n |x_n|_H \le |x|_H$.)

Proposition 1

Let E be finite dimensional (e.g. $E = \mathbb{R}^d$). Then

$$x_n \to x \quad \text{if and only if} \quad x_n \xrightarrow{w} x.$$

Proof. ($\Rightarrow$) In any normed space, if $x_n \to x$ then for each $\ell \in E^*$, $|\ell(x_n) - \ell(x)| \le |\ell|_{E^*} |x_n - x|_E \to 0$, so $x_n \xrightarrow{w} x$.

($\Leftarrow$) Suppose $\dim E = d$ and $x_n \xrightarrow{w} x$. Fix a basis $v_1, \ldots, v_d$ of E with coordinate functionals $\ell_1, \ldots, \ell_d \in E^*$ ($\ell_i(v_j) = \delta_{ij}$), so $y = \sum_{i=1}^d \ell_i(y) v_i$ for every $y \in E$. Weak convergence gives

$\ell_i(x_n) \to \ell_i(x)$ for each i. Since all norms on a finite-dimensional space are equivalent,

$$|x_n - x|_E \le C \sum_{i=1}^{d} |\ell_i(x_n) - \ell_i(x)| \to 0,$$

so $x_n \to x$ in norm.

E.g. For $x^{(n)}, x \in \mathbb{R}^d$ with $\{x^{(n)}\}$ a sequence, $x^{(n)} \to x$ iff $x_i^{(n)} \to x_i$ for every $i = 1, 2, \dots, d$.

Remark 2. In general, $x_n \xrightarrow{w} x$ does *not* imply $x_n \to x$ in norm. For example, let H be an infinite-dimensional Hilbert space with a CONS $\{e_n : n \ge 1\}$. Then for $n \ne m$,

$$|e_n - e_m|_H^2 = |e_n|^2 - 2\langle e_n, e_m\rangle + |e_m|^2 = 2,$$

so $\{e_n\}$ has no norm-convergent subsequence. But $e_n \xrightarrow{w} 0$: for every $x \in H$, by Parseval $\sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2 = |x|^2 < \infty$, hence $|\langle x, e_n\rangle|^2 \to 0$, i.e. $\langle x, e_n\rangle \to 0 = \langle x, 0\rangle$ as $n \to \infty$.

Recall the canonical embedding $E \subseteq (E^*)^* = E^{**}$: every $x \in E$ defines a functional $T_x \in (E^*)^*$ by $T_x(\ell) = \ell(x)$, $\ell \in E^*$. By the Hahn–Banach theorem,

$$\|T_x\| = \sup_{|\ell|_{E^*} \le 1} |T_x(\ell)| = \sup_{|\ell|_{E^*} \le 1} |\ell(x)| = |x|_E,$$

so $x \mapsto T_x$ is an isometry.

Definition. A sequence $\{x_n\}$ in E is *weakly Cauchy* if $\{\ell(x_n)\}$ is Cauchy for all $\ell \in E^*$.

Note. $x_n \xrightarrow{w} x \implies \{x_n\}$ is weakly Cauchy.

Proposition 2

Let E be a normed vector space. Then:

(a) $x_n \xrightarrow{w} x \implies |x|_E \le \liminf_n |x_n|_E$.

(b) If $\{x_n\}$ is weakly Cauchy, then $\{x_n\}$ is bounded: $\sup_n |x_n|_E < \infty$.

(c) Suppose E is a reflexive Banach space. Then every weakly Cauchy sequence in E has a weak limit in E.

Note on reflexive spaces. $E \subseteq (E^*)^* = E^{**}$. The space is *reflexive* when $E = E^{**}$: for all $T \in (E^*)^*$ there exists $x \in E$ such that $T(\ell) = \ell(x)$ for all $\ell \in E^*$.

Proof. *(a)* Let $x_n \xrightarrow{w} x$. By a corollary of the Hahn–Banach theorem, there exists $\ell \in E^*$ with $|\ell|_{E^*} = 1$ and $\ell(x) = |x|_E$. Then

$$|x|_E = \ell(x) = \lim_n \ell(x_n), \qquad |\ell(x_n)| \le |\ell|_{E^*}|x_n|_E = |x_n|_E \text{ for any } n.$$

Hence $|x|_E = \lim_n \ell(x_n) = \liminf_n \ell(x_n) \le \liminf_n |x_n|_E$.

(b) Let $\{x_n\}$ be weakly Cauchy. Then $\sup_n |\ell(x_n)| < \infty$ for all $\ell \in E^*$, because $\{\ell(x_n)\}$ is Cauchy in $\mathbb{R}$, hence convergent and bounded. Consider $T_{x_n} \in (E^*)^*$ defined by $T_{x_n}(\ell) =$

$\ell(x_n)$, $\ell \in E^*$. Then $\sup_n |T_{x_n}(\ell)| = \sup_n |\ell(x_n)| < \infty$ for all $\ell \in E^*$. Since E^* is Banach, by the Banach–Steinhaus theorem (uniform boundedness),

$$\sup_n \|T_{x_n}\| = \sup_n \sup_{|\ell|_{E^*} \le 1} |T_{x_n}(\ell)| = \sup_n \sup_{|\ell|_{E^*} \le 1} |\ell(x_n)| = \sup_n |x_n|_E < \infty.$$

(c) Let $\{x_n\}$ be a weakly Cauchy sequence in a reflexive Banach space E. By (b), $M := \sup_n |x_n|_E < \infty$. Set $T(\ell) = \lim_n \ell(x_n)$; the limit exists since $\{x_n\}$ is weakly Cauchy, and $T : E^* \to \mathbb{R}$ is linear. Now $|\ell(x_n)| \to |T(\ell)|$ and

$$|\ell(x_n)| \le |\ell|_{E^*} |x_n|_E \le M|\ell|_{E^*} \implies |T(\ell)| \le M|\ell|_{E^*}, \quad \ell \in E^*,$$

so $T \in (E^*)^*$. By reflexivity $(E^*)^* = E$, so there exists $x \in E$ with $T(\ell) = \ell(x)$ for all $\ell \in E^*$. Then $\ell(x) = \lim_n \ell(x_n)$ for all $\ell \in E^*$, i.e. $x_n \xrightarrow{w} x$.

Note. "$\lim_n x_n = x$" means (a) $\lim_n x_n$ exists and (b) it equals x. For (a), it helps to know there is a convergent subsequence x_{n_k} — equivalently, that $\{x_n : n \ge 1\}$ is relatively compact.

Fact 1. If E is infinite dimensional, the weak topology on E is not metrizable; in particular, weak convergence cannot be defined by any metric.

Proof. Suppose, for contradiction, that the weak topology is induced by a metric. Then 0 has a countable weak-neighborhood basis $\{V_k\}_{k \ge 1}$. Every weak neighborhood of 0 contains a basic neighborhood

$$W(\ell_1, \dots, \ell_m; \epsilon) = \{x \in E : |\ell_i(x)| < \epsilon,\ i = 1, \dots, m\}, \quad \ell_i \in E^*,\ \epsilon > 0,$$

so each V_k contains such a W_k built from a finite set $F_k \subseteq E^*$. Let $F = \bigcup_k F_k$, a countable subset of E^*.

Given any $\ell \in E^*$, the set $\{x : |\ell(x)| < 1\}$ is a weak neighborhood of 0, hence contains some $V_k \supseteq W_k = W(\phi_1, \dots, \phi_m; \epsilon)$ with $\phi_j \in F_k$. In particular $\bigcap_j \ker \phi_j \subseteq \{x : |\ell(x)| < 1\}$, and by scaling $\bigcap_j \ker \phi_j \subseteq \ker \ell$. By the elementary lemma (if $\bigcap_j \ker \phi_j \subseteq \ker \ell$ then $\ell \in \operatorname{span}\{\phi_1, \dots, \phi_m\}$), we get $\ell \in \operatorname{span}(F_k) \subseteq \operatorname{span}(F)$.

Thus $E^* = \operatorname{span}(F)$ has a countable spanning set, i.e. countable Hamel dimension. But E^* is a Banach space, and by the Baire category theorem an infinite-dimensional Banach space cannot have countable Hamel dimension. Since E is infinite dimensional, so is E^* — a contradiction. Hence the weak topology is not metrizable.